

IRREDUCIBILITY OF PSEUDOVARITIES OF SEMIGROUPS

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ABSTRACT

We show that, for every pseudovariety of groups \mathbf{H} , the pseudovariety $\overline{\mathbf{H}}$, consisting of all finite semigroups all of whose subgroups lie in \mathbf{H} , is irreducible for join and the Mal'cev and semidirect products. This improves results of Margolis-Sapir-Weil and Rhodes-Steinberg. We further announce some related results on join irreducibility.

0 OPERATORS ON PSEUDOVARITIES

1 IRREDUCIBILITY

- Definitions
- Results
- Proofs

- By a *pseudovariety* we mean a class of finite semigroups which is closed under taking homomorphic images (\mathcal{H}), subsemigroups (\mathcal{S}) and finite direct products (\mathcal{P}_{fin}).
- In view of the applications, a key problem on pseudovarieties is the *membership problem*: for a pseudovariety \mathbf{V} , to determine whether a given finite semigroup S belongs to \mathbf{V} .
- If it admits an effective solution, then we say that \mathbf{V} is *decidable*.
- There are too many (2^{\aleph_0}) pseudovarieties for all of them to be decidable.

- A natural way to define pseudovarieties is by specifying a generating class of finite semigroups.
- Because of subcommutation relations between the defining closure operators, the pseudovariety generated by a class \mathcal{C} of finite semigroups is the class $\mathcal{HSP}_{\text{fin}}(\mathcal{C})$.
- Three operators on pseudovarieties have played a central role in the theory of finite semigroups:
 - *join*: $\mathbf{V} \vee \mathbf{W}$ is the pseudovariety generated by $\mathbf{V} \cup \mathbf{W}$;
 - *semidirect product*: $\mathbf{V} * \mathbf{W}$ is the pseudovariety generated by the class of all semidirect products of the form $S * T$ with $S \in \mathbf{V}$ and $T \in \mathbf{W}$;
 - *Mal'cev product*: $\mathbf{V} \textcircled{m} \mathbf{W}$ is the pseudovariety generated by the class of all finite semigroups S for which there is a homomorphism $\varphi : S \rightarrow T$ with $T \in \mathbf{W}$ and $\varphi^{-1}(e) \in \mathbf{V}$ for every idempotent $e \in \varphi(S)$.

- As shown by [Albert-Baldinger-Rhodes'1992] and [Rhodes-Steinberg'1999] (see also [Auinger-Steinberg'2001]), none of the above operators preserves decidability.
- Yet, obtaining a proper decomposition of a pseudovariety in terms of such operators has structural significance and is considered a worthwhile pursuit.
- Sometimes, rather than a finitary decomposition, one looks for a hierarchical decomposition or *filtration* of a given pseudovariety.

Examples:

- Krohn-Rhodes decomposition theory:

$$\mathbf{S} = \bigcup_{n \geq 0} (\mathbf{A} * \mathbf{G})^n * \mathbf{A},$$

where decidability of each positive level remains an open problem.

- Dot-depth hierarchy: this may be viewed as a filtration

$$\mathbf{A} = \bigcup_{n \geq 0} \mathbf{B}_n,$$

where only levels 0 and 1 are known to be decidable. Although several decompositions in terms of operators have been proposed for the levels of the hierarchy, none has been proved in general and several have been disproved.

- Let θ be a binary operator on pseudovarieties and let \mathbf{V} be a pseudovariety.
- We say that \mathbf{V} is *strictly θ -irreducible* if

$$\mathbf{V} = \mathbf{V}_1 \theta \mathbf{V}_2 \implies (\mathbf{V} = \mathbf{V}_1 \vee \mathbf{V} = \mathbf{V}_2).$$

- We say that \mathbf{V} is *(strongly) θ -irreducible* (or *θ -prime*) if

$$\mathbf{V} \subseteq \mathbf{V}_1 \theta \mathbf{V}_2 \implies (\mathbf{V} \subseteq \mathbf{V}_1 \vee \mathbf{V} \subseteq \mathbf{V}_2).$$

- Note that, if $\mathbf{V}_1 \vee \mathbf{V}_2 \subseteq \mathbf{V}_1 \theta \mathbf{V}_2$ for all $\mathbf{V}_1, \mathbf{V}_2$, then θ -irreducibility implies strict θ -irreducibility.
- As has been shown in [Rhodes-Steinberg'2009], the converse fails for the join, for instance for the pseudovarieties $\mathbf{N} = \llbracket x^\omega = 0 \rrbracket$ and $\llbracket xyz = x^2 = 0 \rrbracket$.

THEOREM (JA' 1991)

The pseudovariety \mathbf{J} is strictly join reducible.

THEOREM (MARGOLIS-SAPIR-WEIL' 1998)

If \mathbf{H} is a pseudovariety of groups closed under extensions, then $\overline{\mathbf{H}}$ is strictly Mal'cev product irreducible, whence also strictly irreducible for semidirect product and join.

THEOREM (RHODES-STEINBERG' 2009)

Suppose \mathbf{H} is a pseudovariety of groups containing a non-nilpotent group. Then $\overline{\mathbf{H}}$, $\mathbf{CR} \cap \overline{\mathbf{H}}$ and $\mathbf{DS} \cap \overline{\mathbf{H}}$ are all finite join irreducible.

THEOREM (JA-KLÍMA'2011)

If \mathbf{H} is an arbitrary pseudovariety of groups, then $\overline{\mathbf{H}}$ is Mal'cev product irreducible, whence also irreducible for semidirect product and join.

- This completely solves Problem 47 in the “q-theory” book [Rhodes-Steinberg'2009], the strict version of which had already been proposed in [Margolis-Sapir-Weil'1998].

THEOREM

For every nontrivial pseudovariety of groups H , the pseudovarieties \mathbf{A} , $\overline{\mathbf{H}}$, $\mathbf{DS} \cap \overline{\mathbf{H}}$, and $\mathbf{CR} \cap \overline{\mathbf{H}}$ are join irreducible in the lattice $\mathcal{L}_o(\mathbf{S})$ of pseudovarieties of ordered semigroups.

THEOREM

Let \mathbf{V} be a pseudovariety which is closed under concatenation and contains some nontrivial group. Then \mathbf{V} is join irreducible in $\mathcal{L}_o(\mathbf{S})$.

COROLLARY

*The complexity pseudovarieties $\mathbf{C}_n = (\mathbf{A} * \mathbf{G})^n * \mathbf{A}$ are join irreducible in $\mathcal{L}_o(\mathbf{S})$.*

COROLLARY

Let \mathbf{H} be a nontrivial pseudovariety of groups. Then none of the pseudovarieties \mathbf{A} , \mathbf{C}_n , $\overline{\mathbf{H}}$, $\mathbf{DS} \cap \overline{\mathbf{H}}$, and $\mathbf{CR} \cap \overline{\mathbf{H}}$ can be generated by proper subpseudovarieties of ordered semigroups.

- We are identifying pseudovarieties of semigroups with the pseudovarieties of ordered semigroups they generate.
- Note that this identification is a lattice homomorphism.
- Hence, join irreducibility of a pseudovariety of semigroups \mathbf{V} in $\mathcal{L}_0(\mathbf{S})$ implies that \mathbf{V} is join irreducible as a pseudovariety of semigroups.
- This shows that the corollary of the preceding page solves Problem 43 in the “q-theory” book [Rhodes-Steinberg'2009].

- The following are deduced from our irreducibility theorem in the same way as in [Margolis-Sapir-Weil'1998]:

COROLLARY

For an arbitrary pseudovariety of groups \mathbf{H} , the pseudovariety $\overline{\mathbf{H}}$ is semidirect product irreducible.

PROOF.

This follows from the inclusion $\mathbf{V} * \mathbf{W} \subseteq \mathbf{LV} \circledast \mathbf{W}$ (see [Rhodes-Steinberg'2009, Corollary 4.1.32]).

From the hypothesis and the Mal'cev irreducibility, we deduce that $\overline{\mathbf{H}}$ is contained in at least one of the Mal'cev factors \mathbf{LV} and \mathbf{W} . In the former case, since $\overline{\mathbf{H}}$ is generated by its monoids, it follows that it is contained in \mathbf{V} . □

COROLLARY

If \mathbf{H} is an arbitrary pseudovariety of groups, then $\overline{\mathbf{H}}$ is join irreducible.

PROOF.

It suffices to note that $\mathbf{V} \vee \mathbf{W} \subseteq \mathbf{V} * \mathbf{W}$ and apply the preceding corollary. □

- Recall Reiterman's Theorem, stating that every pseudovariety admits a definition by *pseudoidentities*, i.e., by formal equalities between elements of a free profinite semigroup $\overline{\Omega}_A \mathbf{S}$.
- You may recall a problem from Jean-Eric Pin's book "Varieties of Formal Languages", which is attributed to Margolis:

Show that there is no proper maximal pseudovariety of semigroups.

- Using Reiterman's theorem, the problem can be solved by establishing that, given a *nontrivial* pseudoidentity $u = v$ and a finite semigroup S , there is some nontrivial consequence $u' = v'$ of $u = v$ which holds in S .

- Since every finite semigroup satisfies some pseudoidentity of the form $x^\omega = x^n$, it suffices to find a nontrivial common consequence of $u = v$ and $x^\omega = x^n$.
- Let x be a variable which occurs neither in u nor in v , and consider the pseudoidentity

$$(ux)^\omega(vx)^n = (ux)^n(vx)^\omega. \quad (1)$$

- Note that it holds in both $\llbracket u = v \rrbracket$ and $\llbracket x^\omega = x^n \rrbracket$.
- It remains to show that the pseudoidentity (1) is nontrivial.
- For this purpose, we invoke the result that the free profinite semigroup is equidivisible, meaning that any two factorizations of the same element admit a common refinement [JA-Costa'2009].

- Indeed, if

$$(ux)^\omega (vx)^n = (ux)^n (vx)^\omega$$

is a trivial pseudoidentity, then we have the following two factorizations of the same element of a free profinite semigroup:

$$\begin{aligned} uxux \dots ux \color{red}{ux} (ux)^{\omega-n-1} (vx)^n \\ uxux \dots ux \color{red}{vx} (vx)^{\omega-1} \end{aligned}$$

where the first $n + 1$ occurrences of the factor x are distinguished. Since x occurs in neither u nor in v , a common refinement of the two factorizations must match the distinguished occurrences of x . Hence $u = v$ is trivial, which contradicts the assumption. □

- Margolis' result is a simple consequence of the strict join irreducibility of $\mathbf{S} = \overline{\mathbf{H}}$.
- The above solution applies equally well to an arbitrary pseudovariety of the form $\overline{\mathbf{H}}$ since $\overline{\Omega_A \mathbf{H}}$ is always equidivisible [JA-Costa'2009].

- Let S and T be semigroups and let $f : S^1 \rightarrow T^1$ be an arbitrary function.
- Define on the set

$$M(S, T, f) = S \uplus S^1 \times T^1 \times S^1$$

a multiplication by ($s \in S$, $s_i, s'_i \in S^1$, and $t, t' \in T^1$):

$$s_1 \cdot s_2 = s_1 s_2$$

$$s \cdot (s_1, t, s_2) = (ss_1, t, s_2)$$

$$(s_1, t, s_2) \cdot s = (s_1, t, s_2 s)$$

$$(s_1, t, s_2) \cdot (s'_1, t', s'_2) = (s_1, t f(s_2 s'_1) t', s'_2).$$

LEMMA

The set $M(S, T, f)$ is a semigroup for the above operation. All its subgroups are isomorphic to subgroups of either S or T .

- For pseudovarieties \mathbf{U} and \mathbf{V} , let $\mathbf{U} \bullet \mathbf{V}$ be the pseudovariety generated by all semigroups of the form $M(S, T, f)$, with $S \in \mathbf{U}$ and $T \in \mathbf{V}$.
- In particular, every pseudovariety of the form $\overline{\mathbf{H}}$ is a \bullet -idempotent.

THEOREM (ENCODING)

Let \mathbf{V} be a bullet idempotent and let A and B be finite sets. If $\varphi : B \rightarrow \overline{\Omega}_A \mathbf{V}$ is an injective function, then the unique continuous homomorphism $\psi : \overline{\Omega}_B \mathbf{V} \rightarrow \overline{\Omega}_{A \uplus \{z\}} \mathbf{V}$ such that $\psi(b) = \varphi(b)z$ is injective.

PROOF SKETCH.

- Suppose that $u, v \in \overline{\Omega}_B \mathbf{V}$ are distinct yet $\psi(u) = \psi(v)$.
- Let $T \in \mathbf{V}$ fail $u = v$, say $\tau(u) \neq \tau(v)$ for a continuous homomorphism $\tau : \overline{\Omega}_B \mathbf{V} \rightarrow T$.
- Let $\sigma : \overline{\Omega}_A \mathbf{V} \rightarrow S$ be a continuous homomorphism onto $S \in \mathbf{V}$ such that $\sigma \circ \varphi$ is injective, and let $f : S^1 \rightarrow T^1$ be defined by $f(\sigma(\varphi(b))) = \tau(b)$, and $f(s) = 1$ for all other $s \in S^1$.
- Consider the continuous homomorphism $\zeta : \overline{\Omega}_{A \uplus \{z\}} \mathbf{V} \rightarrow M(S, T, f)$ such that $\zeta(a) = \sigma(a)$ for $a \in A$ and $\zeta(z) = (1, 1, 1)$.
- Then one can check that $\zeta(z\psi(w)) = (1, \tau(w), 1)$ for every $w \in \overline{\Omega}_B \mathbf{V}$.
- Hence, from the assumption that $\psi(u) = \psi(v)$, we deduce that $\tau(u) = \tau(v)$, which contradicts the choice of τ . □

THEOREM

Every bullet idempotent is strictly join irreducible.

- For the only applications we have found of the preceding theorem, we can do better:

THEOREM

For an arbitrary pseudovariety of groups \mathbf{H} , the pseudovariety $\overline{\mathbf{H}}$ is Mal'cev product irreducible.

- Suppose that $\bar{\mathbf{H}} \subseteq \mathbf{V} \textcircled{m} \mathbf{W}$ and that $\bar{\mathbf{H}}$ is contained in neither \mathbf{V} nor \mathbf{W} .
- There are pseudoidentities $u = v$ and $w = t$ which fail in $\bar{\mathbf{H}}$ but hold respectively in \mathbf{V} and \mathbf{W} .
- By a result of [Koryakov'1995], we may assume that $u, v \in \bar{\Omega}_2 \mathbf{S}$.
- Let A be a finite set such that $w, t \in \bar{\Omega}_A \mathbf{S}$ and choose $z \notin A$. Note that $\mathbf{V} \textcircled{m} \mathbf{W}$ satisfies the pseudoidentity

$$\begin{aligned} & u((wz)^\omega (tz)^\omega, (wz)^\omega (tzwz)^\omega (tz)^\omega) \\ & = v((wz)^\omega (tz)^\omega, (wz)^\omega (tzwz)^\omega (tz)^\omega) \end{aligned} \quad (2)$$






- It remains to show that the above pseudoidentity fails in $\bar{\mathbf{H}}$. By a result of [Margolis-Sapir-Weil'1998], the pseudovariety $\bar{\mathbf{H}}$ fails the pseudoidentity







$$u(x^\omega y^\omega, x^\omega (yx)^\omega y^\omega) = v(x^\omega y^\omega, x^\omega (yx)^\omega y^\omega). \quad (3)$$



- Let $p : \bar{\Omega}_{\{x,y\}} \mathbf{S} \rightarrow \bar{\Omega}_{\{x,y\}} \bar{\mathbf{H}}$ and $q : \bar{\Omega}_{AU\{z\}} \mathbf{S} \rightarrow \bar{\Omega}_{AU\{z\}} \bar{\mathbf{H}}$ be the natural continuous homomorphisms.
- To apply the Encoding Theorem, let $\varphi : \{x, y\} \rightarrow \bar{\Omega}_A \bar{\mathbf{H}}$ be the function that maps x to $q(w)$ and y to $q(t)$. Note that φ is injective because the pseudoidentity $w = t$ fails in $\bar{\mathbf{H}}$.
- Let $\psi : \bar{\Omega}_{\{x,y\}} \bar{\mathbf{H}} \rightarrow \bar{\Omega}_{AU\{z\}} \bar{\mathbf{H}}$ be the resulting injective continuous homomorphism defined in the Encoding Theorem.
- Since, for $s \in \bar{\Omega}_{\{x,y\}} \mathbf{S}$,

$$\psi\left(p(s(x^\omega y^\omega, x^\omega (yx)^\omega y^\omega))\right) = q\left(s((wz)^\omega (tz)^\omega, (wz)^\omega (twz)^\omega (tz)^\omega)\right)$$

and ψ is injective, from the fact that the pseudoidentity (3) fails in $\bar{\mathbf{H}}$ it follows that so does (2). \square

-  D. Albert, R. Baldinger, and J. Rhodes, *The identity problem for finite semigroups (the undecidability of)*, J. Symbolic Logic **57** (1992), 179–192.
-  J. Almeida, *On direct product decompositions of finite \mathcal{J} -trivial semigroups*, Int. J. Algebra Comput. **1** (1991), 329–337.
-  _____, *Finite semigroups and universal algebra*, World Scientific, Singapore, 1995, English translation.
-  J. Almeida and A. Costa, *Infinite-vertex free profinite semigroupoids and symbolic dynamics*, J. Pure Appl. Algebra **213** (2009), 605–631.
-  J. Almeida and O. Klíma, *On the irreducibility of pseudovarieties of semigroups*, Tech. Report CMUP 2011-33, Masaryk Univ. and Univ. Porto, 2011.

-  _____, *Representations of relatively free profinite semigroups, irreducibility, and order primitivity*, in preparation.
-  K. Auinger and B. Steinberg, *On the extension problem for partial permutations*, Proc. Amer. Math. Soc. **131** (2003), 2693–2703.
-  I. O. Koryakov, *Embeddings of pseudofree semigroups*, Russian Math. (Iz. VUZ) **39** (1995), 53–59.
-  S. Margolis, M. Sapir, and P. Weil, *Irreducibility of certain pseudovarieties*, Comm. Algebra **26** (1998), 779–792.
-  J.-E. Pin, *Varieties of formal languages*, Plenum, London, 1986, English translation.
-  J. Reiterman, *The Birkhoff theorem for finite algebras*, Algebra Universalis **14** (1982), 1–10.

-  J. Rhodes and B. Steinberg, *Pointlike sets, hyperdecidability and the identity problem for finite semigroups*, Int. J. Algebra Comput. **9** (1999), 475–481.
-  _____, *The q-theory of finite semigroups*, Springer Monographs in Mathematics, Springer, 2009.