

Two-way transducers

On the good use of semigroups

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for many fruitful discussions

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Outline

One-way transducers

- Transducers

- Sequential transducers

- Decomposition theorem

Two-way transducers

- With a one-way output tape

- With a two-way input/output tape

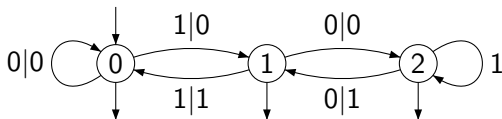
Non-deterministic transducers

Deterministic transducers

Transducers

A **transducer** \mathcal{T} is an automaton with input and output. Each transition is labeled by a pair of words. It **realizes** a **rational** relation $|\mathcal{T}| \subseteq A^* \times B^*$.

Example (Division of integers by 3 in base 2)



For instance

$$0 \xrightarrow{1|0} 1 \xrightarrow{0|0} 2 \xrightarrow{1|1} 2 \xrightarrow{0|1} 1 \xrightarrow{1|1} 0$$

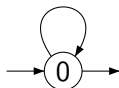
is an accepting path, and $(10101, 00111) = (\langle 21 \rangle_2, \langle 7 \rangle_2)$ is accepted.

Real-time transducers

Equivalent presentations of transducers.

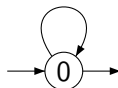
Example

Classical transducer

$$\begin{array}{l} 0|0 \quad 1|1 \\ \varepsilon|0 \quad \varepsilon|1 \end{array}$$


Labels in $(A \times \varepsilon) + (\varepsilon \times B)$

Real-time transducer

$$\begin{array}{l} 0|A^*0A^* \\ 1|A^*1A^* \end{array}$$


Labels in $A \times \text{Rat}(B^*)$

$$R = \{(u, v) \mid u \text{ subword of } v\}$$

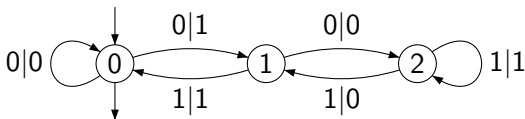
Sequential transducers

A transducer is **left-sequential** (resp. **right-sequential** if its input automaton is deterministic (resp. co-deterministic, that is, deterministic if transitions are reversed).

The previous transducer is left-sequential. The following one is right-sequential.

Example (Multiplication of integers by 3 in base 2)

$$|\mathcal{T}| = \{(\langle n \rangle_2, \langle 3n \rangle_2) \mid n \geq 0\}$$



For instance

$$0 \xrightarrow{0|1} 1 \xrightarrow{0|0} 2 \xrightarrow{1|1} 2 \xrightarrow{1|0} 1 \xrightarrow{1|1} 0$$

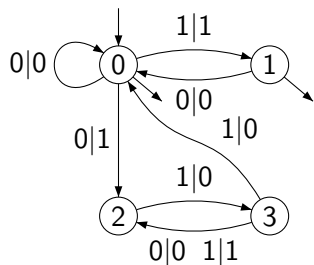
is an accepting path, and $(00111, 10101) = (\langle 7 \rangle_2, \langle 21 \rangle_2)$ is accepted.

Functional transducers

A transducer is **functional** if it realizes a function.

Left and right-sequential transducers are functional but some functional transducers are neither left-sequential nor right-sequential.

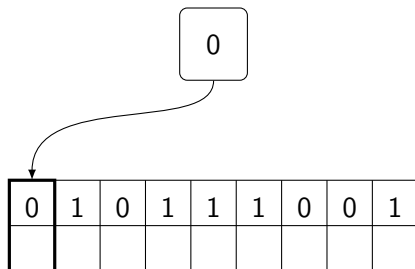
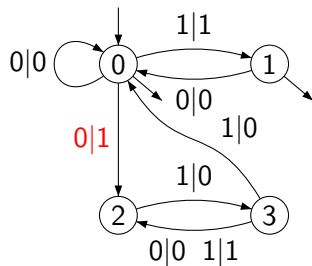
Example (Normalization in Fibonacci base : $011 \rightarrow 100$)



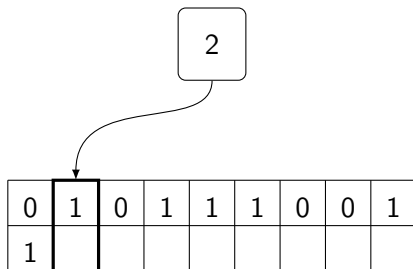
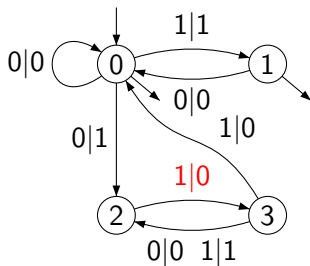
n	55	34	21	13	8	5	3	2	1
61	0	1	0	1	1	1	0	0	1
61	0	1	1	0	0	1	0	0	1
61	1	0	0	0	0	1	0	0	1

$0 \xrightarrow{0|1} 2 \xrightarrow{1|0} 3 \xrightarrow{0|0} 2 \xrightarrow{1|0} 3 \xrightarrow{1|0} 0 \xrightarrow{1|1} 1 \xrightarrow{0|0} 0 \xrightarrow{0|0} 0 \xrightarrow{1|1} 1$

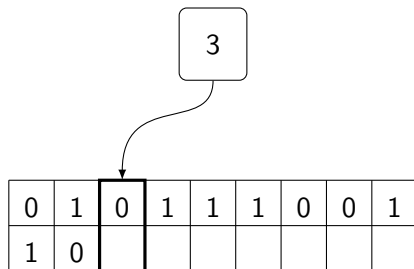
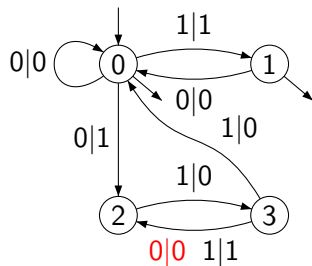
Example



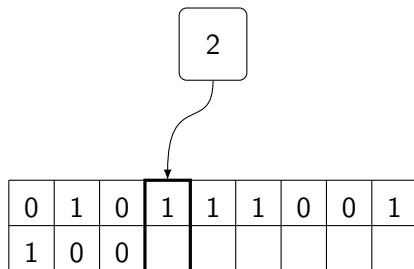
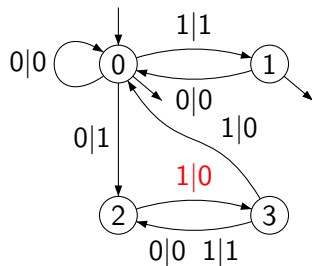
Example



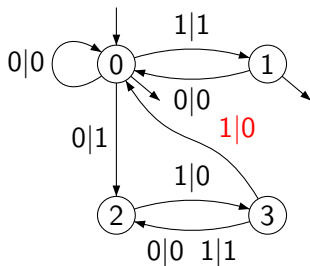
Example



Example



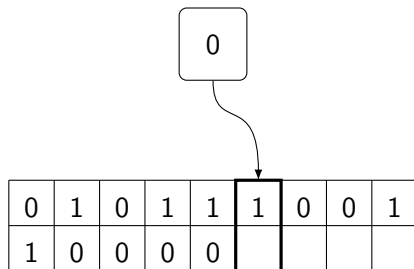
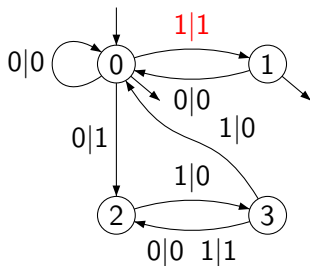
Example



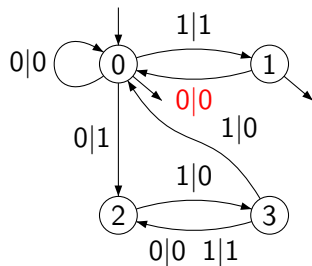
3

0	1	0	1	1	1	0	0	1
1	0	0	0					

Example



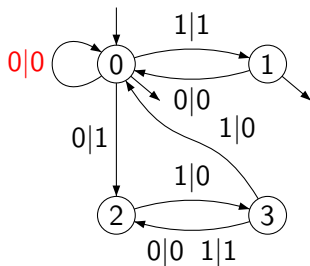
Example



1

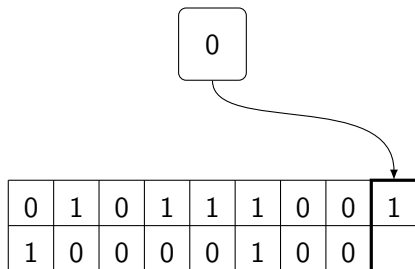
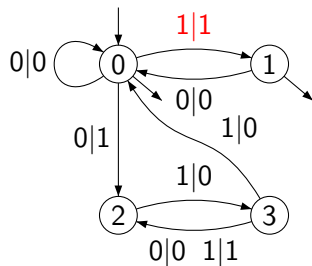
0	1	0	1	1	1	0	0	1
1	0	0	0	0	1			

Example

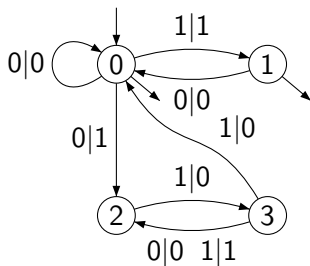


0	1	0	1	1	1	0	0	1
1	0	0	0	0	1	0		

Example



Example



1

0	1	0	1	1	1	0	0	1
1	0	0	0	0	1	0	0	1

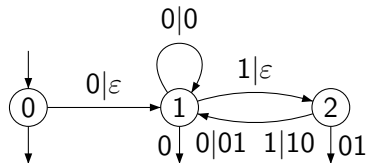
Decomposition theorem

Theorem (Elgot & Mezei)

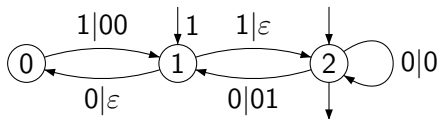
Any rational function is the composition of right-sequential function and a left-sequential function (or a left-sequential function and a right-sequential function).

Example (Normalization in Fibonacci base : $011 \rightarrow 100$)

Left-sequential



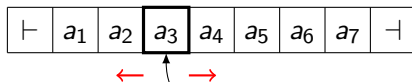
Right-sequential



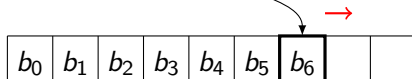
$$\begin{array}{cccccccccccccccc}
 0 & \xrightarrow{0|\varepsilon} & 1 & \xrightarrow{1|\varepsilon} & 2 & \xrightarrow{0|01} & 1 & \xrightarrow{1|\varepsilon} & 2 & \xrightarrow{0|01} & 1 & \xrightarrow{1|\varepsilon} & 2 & \xrightarrow{1|10} & 1 & \xrightarrow{1|\varepsilon} & 2 & \xrightarrow{|01} \\
 |1 & \xrightarrow{0|\varepsilon} & 0 & \xrightarrow{1|00} & 1 & \xrightarrow{0|\varepsilon} & 0 & \xrightarrow{1|00} & 1 & \xrightarrow{1|\varepsilon} & 2 & \xrightarrow{0|0} & 2 & \xrightarrow{0|01} & 1 & \xrightarrow{1|\varepsilon} & 2 &
 \end{array}$$

With a one-way output tape

Two-way
input tape



One-way
output tape



Transitions $p \xrightarrow{a|K,d} q$ for $a \in A$, $K \in \text{Rat}(B^*)$ and $d \in \{\triangleleft, \triangleright\}$.

Example

$\vdash|\epsilon, \triangleright$

$a|a, \triangleright$

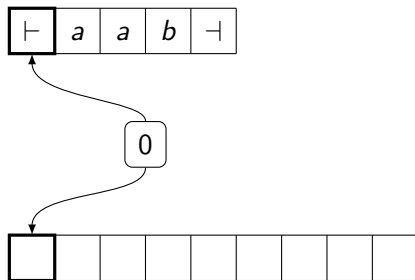
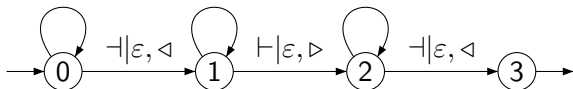
$b|b, \triangleright$

$a|\epsilon, \triangleleft$

$b|\epsilon, \triangleleft$

$a|a, \triangleright$

$b|b, \triangleright$



Example

$\vdash|\epsilon, \triangleright$

$a|a, \triangleright$

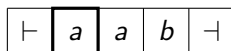
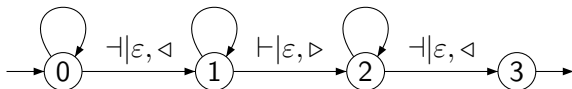
$b|b, \triangleright$

$a|\epsilon, \triangleleft$

$b|\epsilon, \triangleleft$

$a|a, \triangleright$

$b|b, \triangleright$



0



Example

$\vdash|\epsilon, \triangleright$

$a|a, \triangleright$

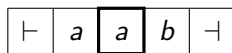
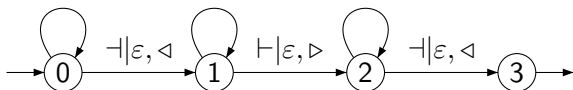
$b|b, \triangleright$

$a|\epsilon, \triangleleft$

$b|\epsilon, \triangleleft$

$a|a, \triangleright$

$b|b, \triangleright$



0



Example

$\vdash|\epsilon, \triangleright$

$a|a, \triangleright$

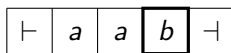
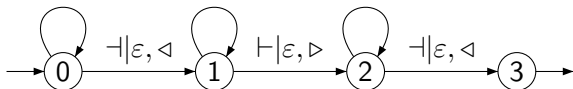
$b|b, \triangleright$

$a|\epsilon, \triangleleft$

$b|\epsilon, \triangleleft$

$a|a, \triangleright$

$b|b, \triangleright$



0



Example

$\vdash|\epsilon, \triangleright$

$a|a, \triangleright$

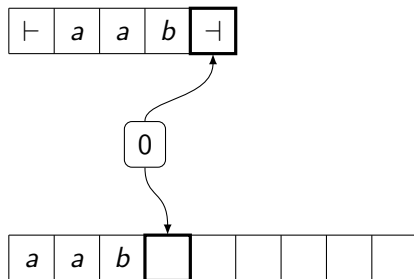
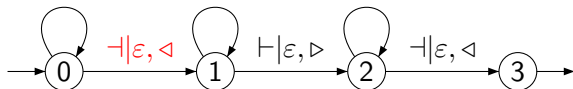
$b|b, \triangleright$

$a|\epsilon, \triangleleft$

$b|\epsilon, \triangleleft$

$a|a, \triangleright$

$b|b, \triangleright$



Example

$\vdash|\epsilon, \triangleright$

$a|a, \triangleright$

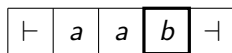
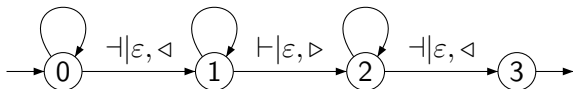
$b|b, \triangleright$

$a|\epsilon, \triangleleft$

$b|\epsilon, \triangleleft$

$a|a, \triangleright$

$b|b, \triangleright$



Example

$\vdash|\epsilon, \triangleright$

$a|a, \triangleright$

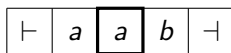
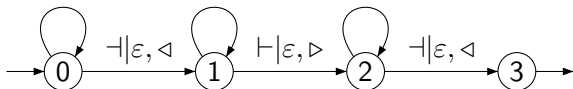
$b|b, \triangleright$

$a|\epsilon, \triangleleft$

$b|\epsilon, \triangleleft$

$a|a, \triangleright$

$b|b, \triangleright$



1



Example

$\vdash|\epsilon, \triangleright$

$a|a, \triangleright$

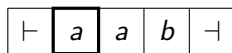
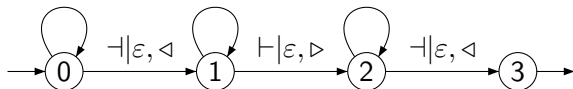
$b|b, \triangleright$

$a|\epsilon, \triangleleft$

$b|\epsilon, \triangleleft$

$a|a, \triangleright$

$b|b, \triangleright$



1



Example

$\vdash|\epsilon, \triangleright$

$a|a, \triangleright$

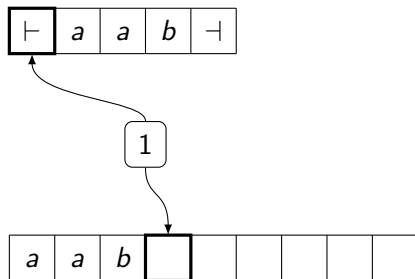
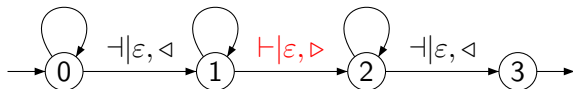
$b|b, \triangleright$

$a|\epsilon, \triangleleft$

$b|\epsilon, \triangleleft$

$a|a, \triangleright$

$b|b, \triangleright$



Example

$\vdash|\epsilon, \triangleright$

$a|a, \triangleright$

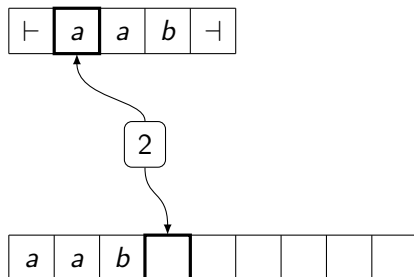
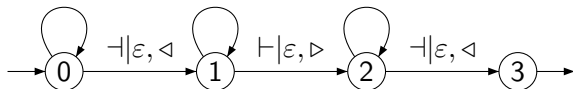
$b|b, \triangleright$

$a|\epsilon, \triangleleft$

$b|\epsilon, \triangleleft$

$a|a, \triangleright$

$b|b, \triangleright$



Example

$\vdash|\epsilon, \triangleright$

$a|a, \triangleright$

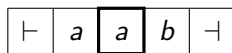
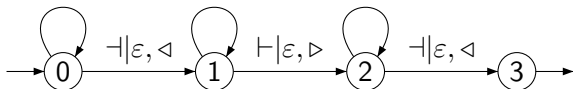
$b|b, \triangleright$

$a|\epsilon, \triangleleft$

$b|\epsilon, \triangleleft$

$a|a, \triangleright$

$b|b, \triangleright$



2



Example

$\vdash|\epsilon, \triangleright$

$a|a, \triangleright$

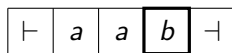
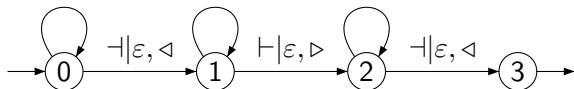
$b|b, \triangleright$

$a|\epsilon, \triangleleft$

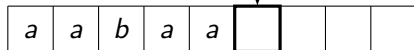
$b|\epsilon, \triangleleft$

$a|a, \triangleright$

$b|b, \triangleright$



2



Example

$\vdash|\epsilon, \triangleright$

$a|a, \triangleright$

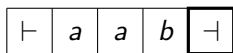
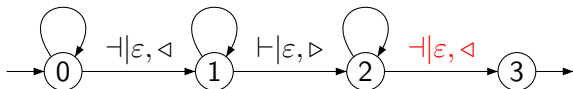
$b|b, \triangleright$

$a|\epsilon, \triangleleft$

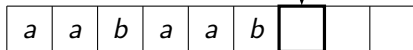
$b|\epsilon, \triangleleft$

$a|a, \triangleright$

$b|b, \triangleright$



2



Example

$\vdash|\epsilon, \triangleright$

$a|a, \triangleright$

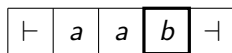
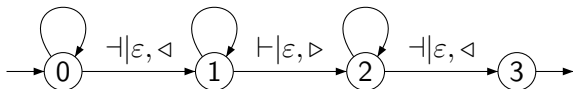
$b|b, \triangleright$

$a|\epsilon, \triangleleft$

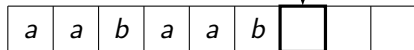
$b|\epsilon, \triangleleft$

$a|a, \triangleright$

$b|b, \triangleright$



3



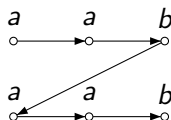
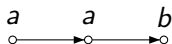
MSO and deterministic transducers

Theorem (Engelfriet & Hoogeboom)

The functions realized by deterministic two-way transducers (with a one-way output tape) are exactly MSO definable functions.

MSO definable function means here that the image (seen as a graph) of a word w , is interpreted in k copies of w .

Example (Function $w \mapsto ww$)



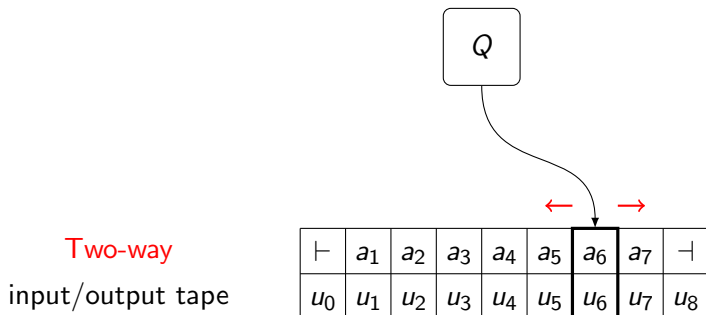
$$Q'_a(x, i) \equiv Q_a(x)$$

$$Q'_b(x, i) \equiv Q_b(x)$$

$$\text{succ}'((x, i), (y, j)) \equiv \text{succ}(x, y) \wedge i = j \quad \vee$$

$$(x, i) = (\max, 0) \wedge (y, j) = (\min, 1)$$

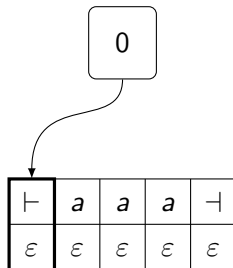
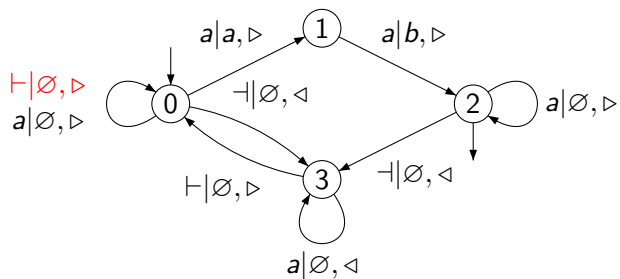
With a two-way input/output tape



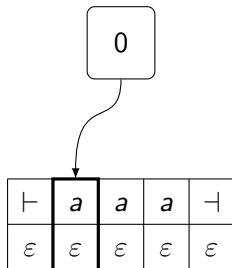
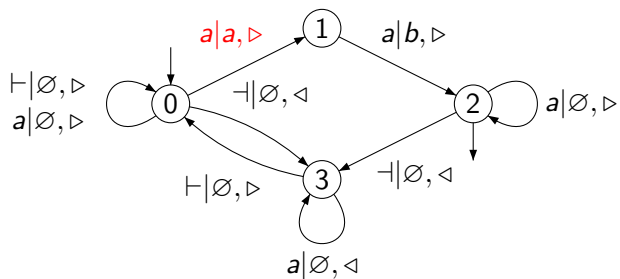
Transitions $p \xrightarrow{a|K,d} q$ for $a \in A$, $K \in \text{Rat}(B^*)$ and $d \in \{\triangleleft, \triangleright\}$.

- ▶ If $K \neq \emptyset$, the transition writes some word $u \in K$ to the corresponding cell of the output tape.
- ▶ If $K = \emptyset$, it writes nothing: the output tape is not modified.

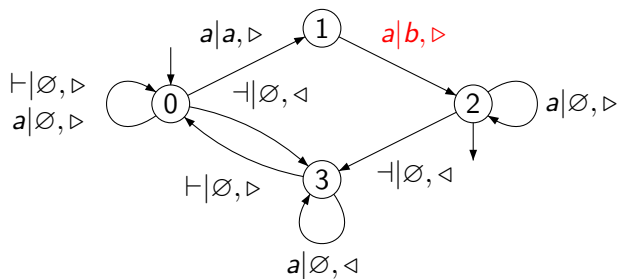
Example



Example

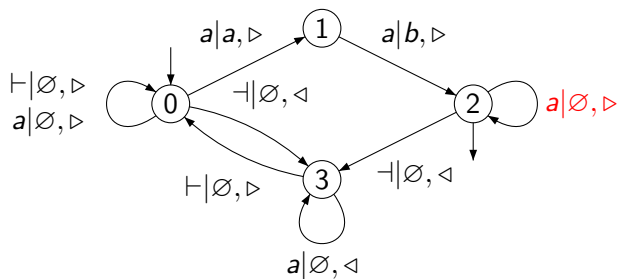


Example



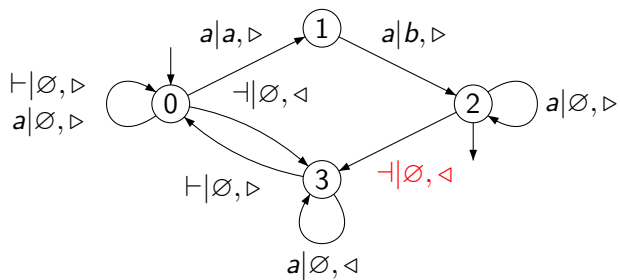
\vdash	a	a	a	\neg
ε	a	ε	ε	ε

Example



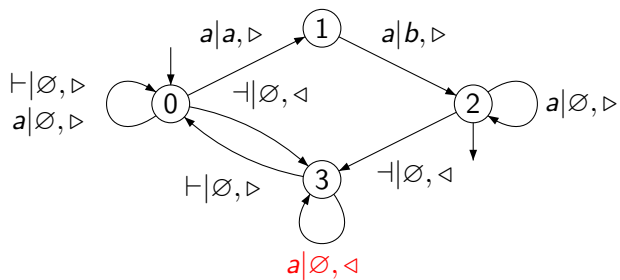
\top	a	a	a	\neg
ϵ	a	b	ϵ	ϵ

Example



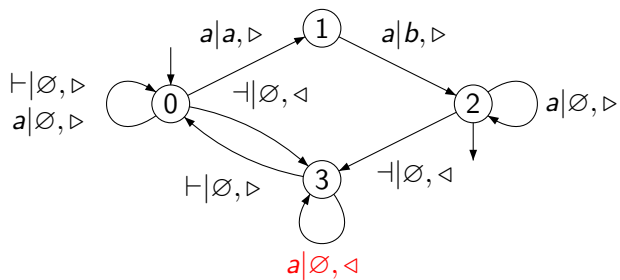
⊢	a	a	a	¬
ε	a	b	ε	ε

Example



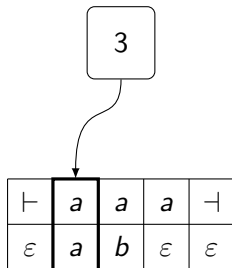
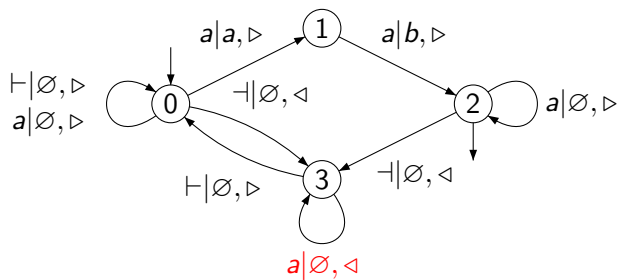
\vdash	a	a	a	\neg
ε	a	b	ε	ε

Example

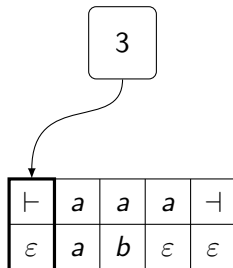
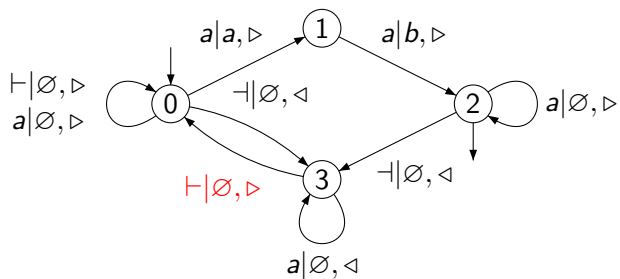


\vdash	a	a	a	\neg
ε	a	b	ε	ε

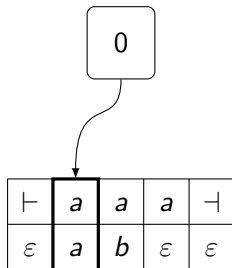
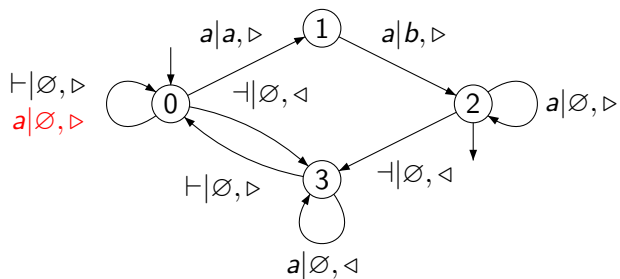
Example



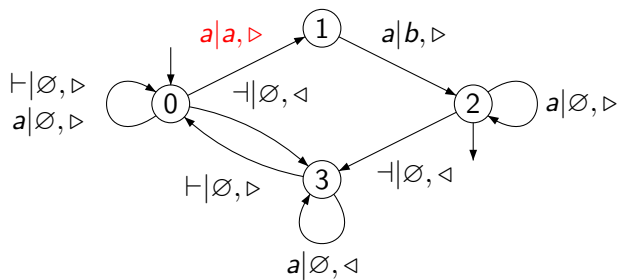
Example



Example

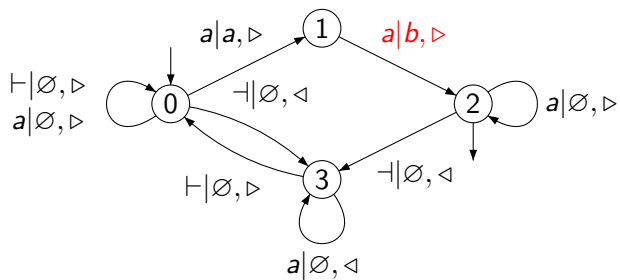


Example



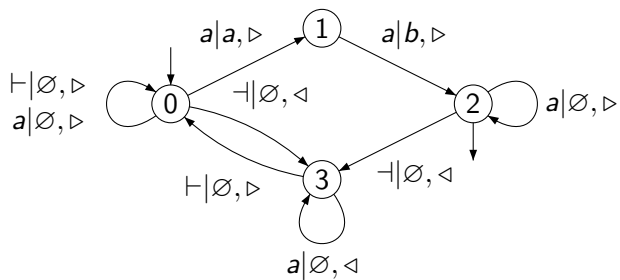
\vdash	a	a	a	\neg
ε	a	b	ε	ε

Example



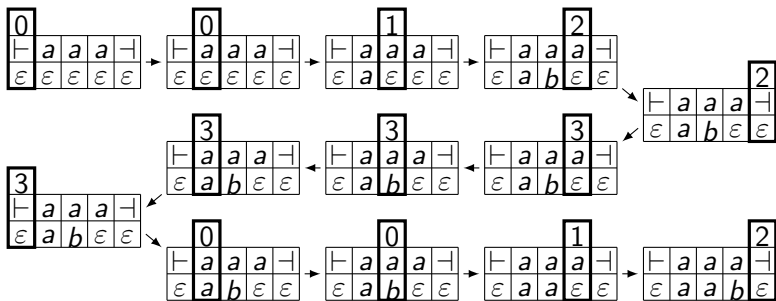
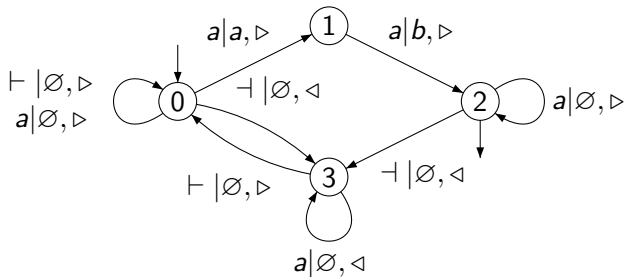
\vdash	a	a	a	\neg
ε	a	a	ε	ε

Example



⊢	a	a	a	¬
ε	a	a	b	ε

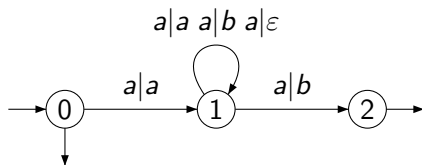
Example



Rationality

Theorem

Any two-way transducer with a two-way output tape is equivalent to a one-way transducer. The relation that it realizes is thus rational.

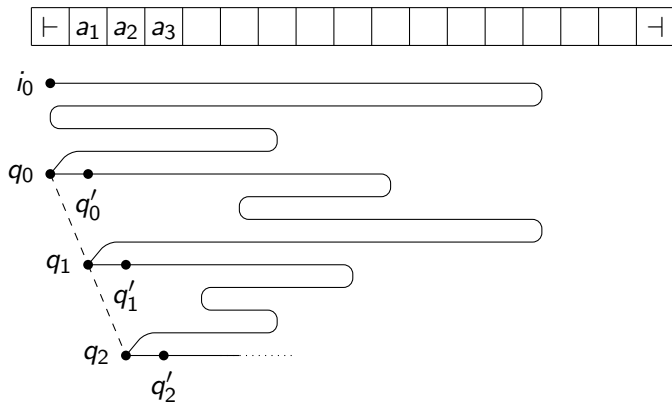


Proof of an easy special case

We suppose

- ▶ Each transition always outputs something ($K \neq \emptyset$).
- ▶ Each accepting run ends on the right end marker \dashv .

One considers the following decomposition of a run



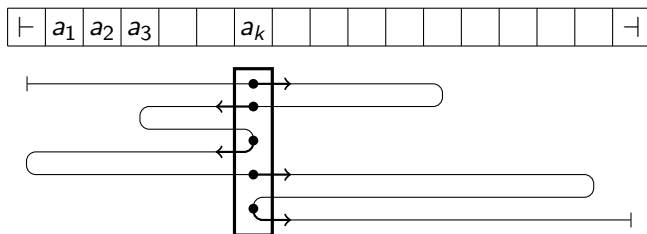
where the last visit of each position i is made in state q_i .

Construction

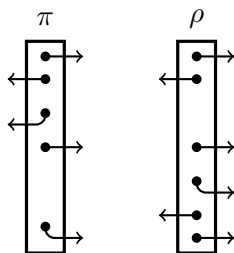
The one-way transducer

- ▶ successively guesses the states $q_0, q'_0, q_1, q'_1, \dots$, and the transitions $q_i \xrightarrow{a_i | K_i, 1} q'_i$
- ▶ launch a copy of an automaton to verify that there is indeed a path from q'_i to q_{i+1} in the word $a_{i+1}a_{i+2} \cdots a_n$.

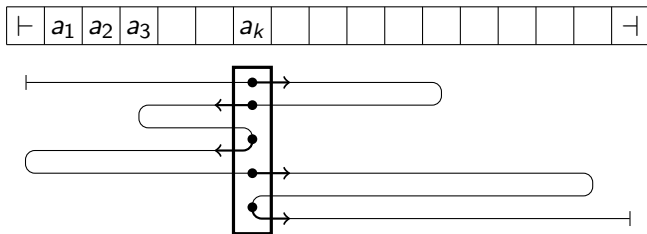
General case: slices of the computation



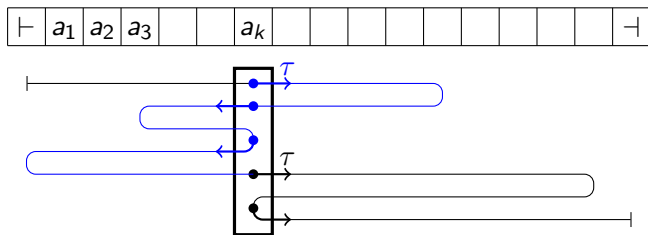
Slice π : sequence τ_1, \dots, τ_n of transitions with the same symbol a .
Write $\pi \rightarrow \rho$ if slices ρ and π can be **consecutive**.



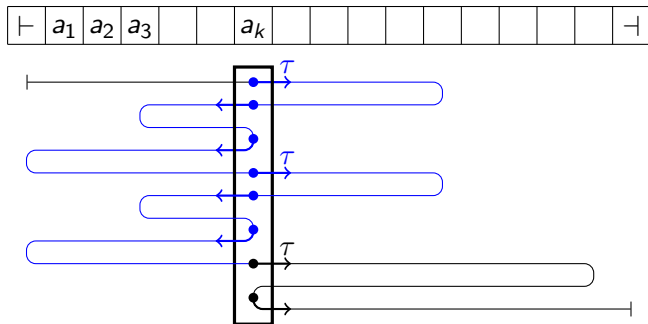
Duplication



Duplication



Duplication



Theorem of Green and Rees

Theorem (Green & Rees)

Let E be a finite alphabet and let \sim the smallest congruence on E^ such that $w \sim w^2$ for each $w \in E^*$. Then \sim has a finite index.*

- ▶ $E = \{\sigma\}$
2 classes: $[\varepsilon]$ and $[\sigma]$
- ▶ $E = \{\sigma, \tau\}$
7 classes: $[\varepsilon]$, $[\sigma]$, $[\tau]$, $[\sigma\tau]$, $[\sigma\tau\sigma]$, $[\tau\sigma]$ and $[\tau\sigma\tau]$
- ▶ $E = \{\sigma, \tau, \xi\}$
120 classes

Compatibility lemma

Let $[\pi]$ be the class of a slice π for \sim .

Define $[\pi] \rightarrow [\rho]$ if $\pi' \rightarrow \rho'$ for some $\pi' \sim \pi$ and $\rho' \sim \rho$.

Lemma

Let $\pi_0 \in E_{a_0}^*, \dots, \pi_n \in E_{a_n}^*$ such that $[\pi_0] \rightarrow [\pi_1] \rightarrow \dots \rightarrow [\pi_n]$.

There exist π'_0, \dots, π'_n such that $\pi'_i \sim \pi_i$ for $0 \leq i \leq n$ and $\pi'_0 \rightarrow \pi'_1 \rightarrow \dots \rightarrow \pi'_n$.

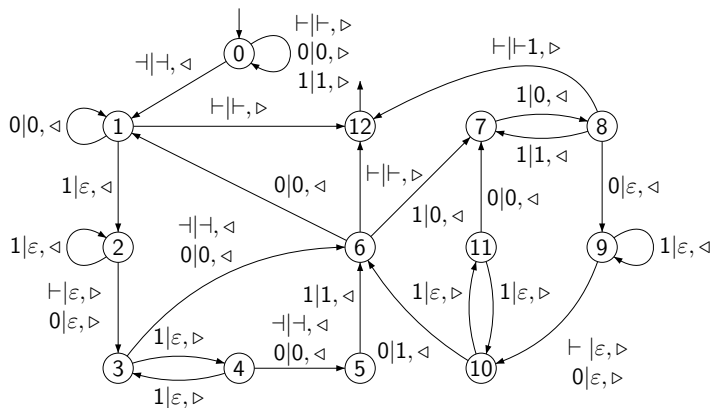
$$\begin{array}{ccccccc} \pi'_0 & \rightarrow & \pi'_1 & \rightarrow & \dots & \rightarrow & \pi'_n \\ \wr & & \wr & & & & \wr \\ [\pi_0] & \rightarrow & [\pi_1] & \rightarrow & \dots & \rightarrow & [\pi_n] \end{array}$$

Deterministic transducers

A deterministic two-way transducer realizes a rational function.
Conversely

Theorem

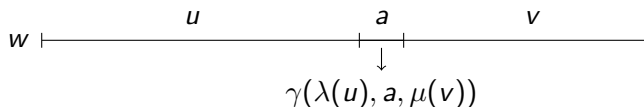
Any rational function can be realized by a deterministic two-way transducer with a two-way output tape.



Bimachines

Monoidal bimachine

- ▶ a morphism $\lambda : A^* \rightarrow M$
- ▶ a morphism $\mu : A^* \rightarrow N$
- ▶ partial function $\gamma : M \times A \times N \rightarrow B^*$.



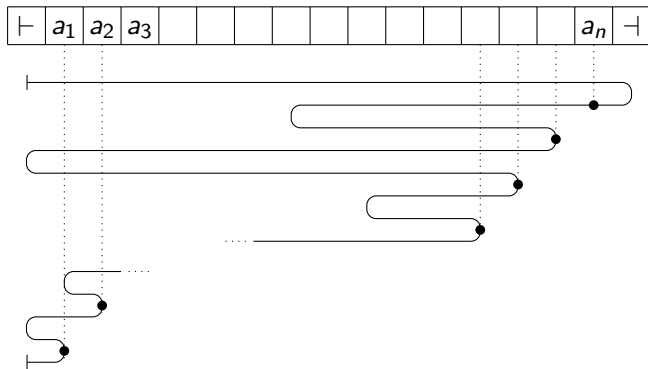
A bimachine **realizes** a function f as follows:

$$f(a_1 \cdots a_n) = \gamma(1_M, a_1, \mu(a_2 \cdots a_n)) \gamma(\lambda(a_1), a_2, \mu(a_3 \cdots a_n)) \\ \cdots \gamma(\lambda(a_1 \cdots a_{n-2}), a_{n-1}, \mu(a_n)) \gamma(\lambda(a_1 \cdots a_{n-1}), a_n, 1_N)$$

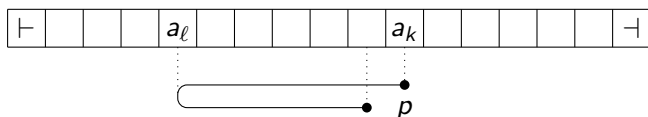
Theorem (Eilenberg)

Any partial rational function $f : A^ \rightarrow B^*$ such that $f(\varepsilon) = \varepsilon$ is realized by a bimachine.*

Global shape of the run of the two-way transducer



Local shape of the run



States $p = (\lambda(a_1 \cdots a_{k-1}), \mu(a_{k+1} \cdots a_n))$

$q = (\lambda(a_1 \cdots a_{k-2}), \mu(a_k \cdots a_n))$

ℓ is the greatest ℓ such that $\lambda(a_\ell \cdots a_{k-2}) \mathcal{R} \lambda(a_\ell \cdots a_{k-1})$

► Finding ℓ

Compute $\lambda(a_{\ell+1} \cdots a_{k-2})$ while moving to the left

► Computing $\lambda(a_1 \cdots a_{k-2})$

Since $\lambda(a_\ell \cdots a_{k-2}) \mathcal{R} \lambda(a_\ell \cdots a_{k-1})$, there exists $m \in M$ such that $\lambda(a_\ell \cdots a_{k-2}) = \lambda(a_\ell \cdots a_{k-1})m$. Then

$\lambda(a_1 \cdots a_{k-2}) = \lambda(a_1 \cdots a_{k-1})m$.

► Coming back to position $k-1$

By definition of ℓ , $\lambda(a_{\ell+1} \cdots a_{k-2}) >_{\mathcal{R}} \lambda(a_{\ell+1} \cdots a_{k-1})$ and thus $\lambda(a_{\ell+1} \cdots a_j) \neq \lambda(a_{\ell+1} \cdots a_{k-1})$ for any $\ell < j < k-1$.