

Profinite semigroups and separation

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ANR FREC, November 27, 2013

Outline

A. Profinite semigroups: introduction / reminders.

B. Separation.

Profinite topologies

Birkhoff, Moore–Smith convergence in general topology (1937)

3 examples of profinite topologies, defined by congruences on algebras:

- ▶ p -adic numbers.
- ▶ Topologies for free groups.
- ▶ Stone's duality of Boolean algebras and Boolean topological spaces (Hausdorff, compact, with a clopen basis).

Compactness of the completion of the topological algebra follows from the fact that congruences have **finite index**.

Semigroups, monoids and groups

- ▶ **Semigroup**: set equipped with an associative binary operation.
- ▶ **Monoid**: semigroup with a unit element.
- ▶ **Group**: monoid in which each element has an inverse.
- ▶ X denotes a finite alphabet.
- ▶ X^+ , $[X^*]$: free semigroup [monoid] on X .

Birkhoff varieties

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Theorem (Birkhoff, 1935)

Varieties are exactly equational classes.

The case of finite semigroups

- ▶ Many interesting **classes of languages**, defined by logical or combinatorial properties, correspond to **classes of finite semigroups**.
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- ▶ Many interesting **classes of languages**, defined by logical or combinatorial properties, correspond to **classes of finite semigroups**.
- ▶ Eilenberg-like correspondence between classes of languages and classes of finite semigroups.
- ▶ **Can we capture these classes by identities?**
- ▶ **What are the properties of these classes?**

Pseudovarieties

Several types of semigroup classes are obtained. We focus on **pseudovarieties**.

- ▶ **Pseudovariety**: class of finite semigroups closed under
 - ▶ finite direct products,
 - ▶ subsemigroup,
 - ▶ quotient.

- ▶ **S**: all finite semigroups.
- ▶ **G**: all finite **groups**.
- ▶ **A**: all finite **aperiodic** (group-free) semigroups.
- ▶ **V**: a generic pseudovariety.

High-level vs. algebraic properties

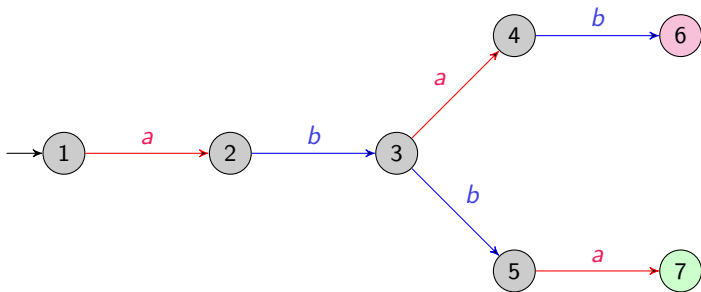
Language L	$\mathcal{A}_{\min}(L)$	$Synt(L)$	Logical definability
Star-free	Counter-free	A	$FO(<)$, LTL
[Sch65,McN-P71,K68]			
Piecewise testable	Very weak + ...	J	$Bool(\Sigma_1)$
[S75]			
$\bigoplus_f \Pi$ non ambiguous	2-way part. ord.	DA	$FO_2(<)$, UTL , $\Sigma_2 \cap \Pi_2$
[Sch76,SchThV01,ThW98,EVW97,PW97]			
Loc. threshold testable	Forbidden patterns	ACom * LI	$FO(\leq)$
[S85,ThW85,BP89]			

Birkhoff's theorem fails

- ▶ **Fact:** given a set of identities, the class of finite semigroups satisfying all the identities of the set is indeed a pseudovariety.
- ▶ **But:** some pseudovarieties are not equational: eg, **G**, **A**.
- ▶ **Remedy:** **Topology!** Instead of considering word identities, replace the free semigroup X^+ by a suitable topological completion.

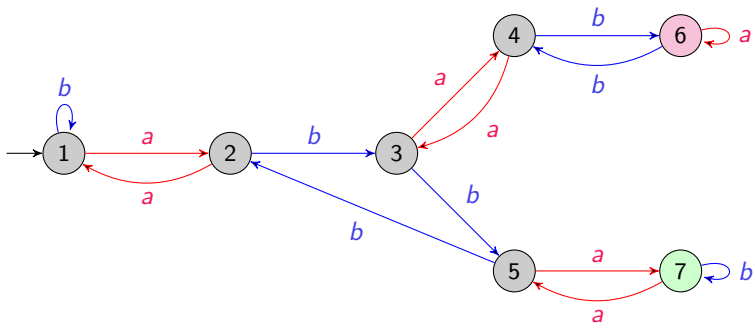
Separating *abab* and *abba* with a finite group

(1) Build an automaton with the two words [courtesy of J.-É. Pin]



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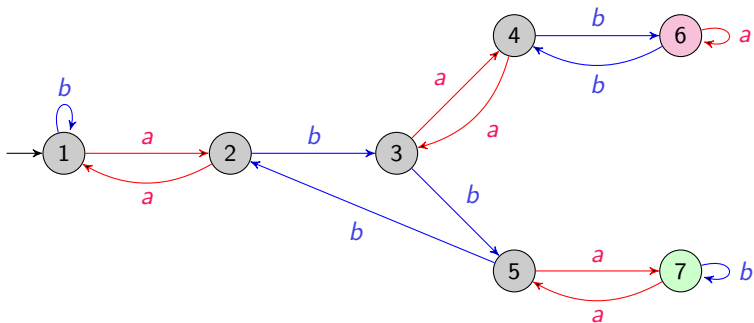
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Separating $abab$ and $abba$ with a finite group

(1) Build an automaton with the two words [courtesy of J.-É. Pin]



(2) Complete into **permutations**. The resulting permutation group separates $abab$ and $abba$ since $1 \cdot abab = 5$ and $1 \cdot abba = 7$.

A metric on X^+

- ▶ X : fixed finite alphabet.
- ▶ A semigroup T separates $u, v \in X^+$ if there is a homomorphism $\varphi : X^+ \rightarrow T$ such that $\varphi(u) \neq \varphi(v)$.
- ▶ Define a pseudo-metric d :

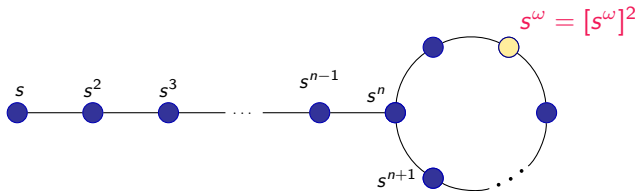
$$\begin{cases} r(u, v) &= \min\{|T| : T \text{ separates } u \text{ and } v\}. \\ d(u, v) &= 2^{-r(u, v)}. \end{cases}$$

The free profinite semigroup

Reutenauer 1979

For $x \in X^+$, the sequence $x^{n!}$ is a Cauchy sequence since it evaluates to $\varphi(x^{n!}) = \varphi(x)^{n!} = \varphi(x)^\omega$ in all semigroup T with less than n elements.

Does not converge in X^+ .



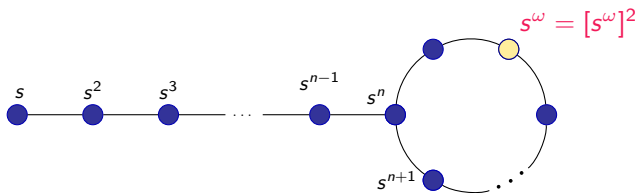
- ▶ The space (X^+, d) is not complete.
- ▶ Free profinite semigroup $\hat{F}_X S$: topological completion of (X^+, d) .
- ▶ Elements of $\hat{F}_X S$ are called **pseudowords**.

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Does not converge in X^+ . It converges in $\hat{F}_X S$ to an idempotent noted x^ω .



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Main properties of the free profinite semigroup

- ▶ The metric d can be extended as usual by continuity to $\hat{F}_X S$.
- ▶ A sequence of words converges in the metric space $(\hat{F}_X S, d)$ iff its projection on every finite semigroup is **ultimately constant**.
- ▶ $(\hat{F}_X S, d)$ is a **compact** space.
- ▶ **Universal** property:

$$\begin{array}{ccc} X^+ & \xrightarrow{\iota} & \hat{F}_X S \\ & \searrow \varphi & \\ & & T \end{array}$$

- ▶ Every $w \in \hat{F}_X S$ can be evaluated in an X -generated semigroup.

$$\begin{aligned} \varphi : X &\longrightarrow T, \\ w_\varphi &= \hat{\varphi}(w). \end{aligned}$$

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Equational description of pseudovarieties

- ▶ A finite semigroup T satisfies $u = v$, for u, v elements of the free profinite semigroup, if for every homomorphism $\varphi : X^+ \rightarrow T$, we have $\hat{\varphi}(u) = \hat{\varphi}(v)$.

Theorem (Reiterman, 1982)

Pseudovarieties are exactly classes defined by identities in elements of the free profinite semigroup.

Examples

- ▶ **S**: $x = x$.
- ▶ **Com**: $xy = yx$.
- ▶ **G**: $x^\omega y = yx^\omega = y$ (abbreviated $x^\omega = 1$).
- ▶ **A**: $x^{\omega+1} = x^\omega$.

Example: commutative p -groups

- ▶ A finite group is a p -group if it has p^k elements for some k .
- ▶ The pseudovariety of commutative p -groups cannot be defined by identities in terms using ω -power.
- ▶ Assume it is defined by such identities $u = v$, plus $xy = yx, x^\omega = 1$.

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- ▶ The pseudovariety of commutative p -groups cannot be defined by identities in terms using ω -power.
- ▶ Assume it is defined by such identities $u = v$, plus $xy = yx, x^\omega = 1$.
- ▶ In $u = v$, all ω -powers can be replaced by 1. So such an identity is equivalent to a word identity.
- ▶ If $|u|_a = |v|_a$ for all letters a , then $u = v$ is not useful (consequence of $xy = yx$).
- ▶ Otherwise, $u = v$ fails in some p -group $\mathbb{Z}/p^k\mathbb{Z}$ for k large enough.

Example: unary alphabets

- ▶ If $X = \{a\}$, the pseudowords which are not words form a **group**.
- ▶ It suffices to show that any two elements are comparable for the prefix and suffix quasi-ordering.
- ▶ Let $u, v \in \hat{F}_X S$: $u = \lim a^{k_n}$ and $v = \lim a^{\ell_n}$, with $k_n, \ell_n \rightarrow \infty$.
- ▶ Since a^{k_p} is prefix and suffix of all a^{ℓ_n} for n large enough, at the limit u is a prefix and suffix of v .
- ▶ Note: the group is infinite (it contains $a^{\omega+k}$ for all $k \in \mathbb{Z}$) and even uncountable.

The free pro- V semigroup on X

- ▶ X : fixed finite alphabet.
- ▶ A semigroup T separates $u, v \in X^+$ if there is a homomorphism $\varphi : X^+ \rightarrow T$ such that $\varphi(u) \neq \varphi(v)$.
- ▶ Define a pseudo-metric d_V :

$$\begin{cases} r_V(u, v) &= \min\{|T| : T \in V \text{ and } T \text{ separates } u \text{ and } v\}. \\ d_V(u, v) &= 2^{-r_V(u, v)}. \end{cases}$$

- ▶ $u \sim_V v$ if and only if $d_V(u, v) = 0$ defines a congruence.
- ▶ Relatively V -free profinite semigroup $\hat{F}_X V$: completion of $(X^+ / \sim_V, d_V)$.

Example: nilpotent semigroups

- ▶ A semigroup is **nilpotent** if it has a unique idempotent which is a zero.
- ▶ Example: Given k , $N_k = \bigcup_{i < k} X^i \cup \{0\}$, and

$$x \cdot y = \begin{cases} 0 & \text{if } |xy| \geq k \\ xy & \text{otherwise} \end{cases}$$

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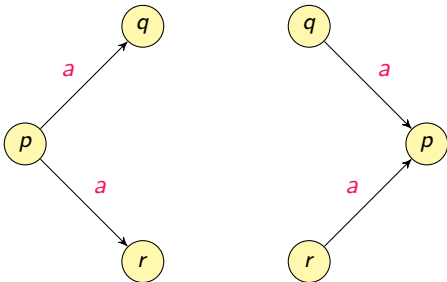
- ▶ N_k separates any two different words of length $< k$.
- ▶ Let \mathbf{N} : pseudovariety of nilpotent semigroups defined by $x^\omega = 0$.
- ▶ Let $u \in \hat{F}_X \mathbf{N}$: $u = \lim_{n \rightarrow \infty} u_n$ where u_n are words. If $|u_n|$ is bounded: u is a word. Otherwise, one can assume $(|u_n| \rightarrow \infty)$, so u evaluates to 0 in every finite nilpotent semigroup.
- ▶ $\hat{F}_A \mathbf{N} = X^+ \cup \{0\}$.

Example: semilattices

- ▶ A semilattice is a commutative and idempotent semigroup: it satisfies $x = x^2$ and $xy = yx$.
- ▶ Example: $(\mathcal{P}(X), \cap)$.
- ▶ Let **SI** the pseudovariety of semilattices. Then $\hat{F}_X \text{SI} \approx (\mathcal{P}(X), \cap)$.

Reversible automata

An automaton is **reversible** if, forgetting its initial and final states, it is deterministic and co-deterministic. **Several initial** (resp. **final**) **states** allowed.



Forbidden configurations in a reversible automaton.

- ▶ Reversible language: recognized by a reversible automaton.
- ▶ **Example**: every finite language is reversible.
- ▶ Can we decide if a language is reversible?

Reversible languages: decision procedure

Theorem (Pin, 1987)

- ▶ A language is reversible if
 1. it is **closed in X^* in the pro- G topology**, and
 2. the idempotents of its syntactic monoid commute.

Clearly, 2 is decidable. Can we also decide 1?

Computing closures

Theorem (Almeida)

If $L \subseteq X^+$ is recognized by a $\varphi : X^+ \rightarrow T \in \mathbf{V}$, then

- ▶ the closure of L in $\hat{F}_X \mathbf{V}$ is equal to $\hat{\varphi}^{-1}(\varphi(L))$,
- ▶ the closure of L in $\hat{F}_X \mathbf{V}$ is open.

The closures of \mathbf{V} -recognizable languages form a basis of the topology of $\hat{F}_X \mathbf{V}$.

Consequence

For two regular languages, their **closures do not intersect** iff the languages can be **separated by a \mathbf{V} -recognizable** language.

The Pin-Reutenauer algorithm

- ▶ We have $X^* \leftrightarrow FG(X) \leftrightarrow \hat{F}_X G$.
- ▶ One can compute the closure of L in $\hat{F}_X G$, or its intersection with $FG(X)$ or with X^* .
- ▶ For groups, it can be shown that the closures intersect in $\hat{F}_X G$ if and only if they intersect in $FG(X)$.
- ▶ The Pin-Reutenauer algorithm computes closures of regular languages in the free group $FG(X)$ or in X^* .

The Pin-Reutenauer algorithm for groups

- ▶ In the free group $FG(X)$ endowed with the pro- G topology, for $K, L \subseteq X^+$ regular:

$$\begin{aligned} \text{cl}_{FG(X)}(KL) &= \text{cl}_{FG(X)}(K) \cdot \text{cl}_{FG(X)}(L), \\ \text{cl}_{FG(X)}(L^*) &= \langle L \rangle \end{aligned}$$

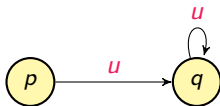
- ▶ Conjectured by Pin and Reutenauer, reduced to another conjecture proved by Ribes and Zalesskiĭ.
- ▶ Equivalent to Rhodes' type II conjecture, proved by Ash.
- ▶ Gives an algorithm to **effectively compute** the closure of a rational language (in the free group, or the free monoid) for the topology d_G .

Closure of a regular language

Theorem (Pin 1991, Pin-Reutenauer 1991)

The closure of a **regular** language for d_G in X^* is still **regular** and can be computed effectively.

Intuitively, the only rule to apply is $\lim_{n \rightarrow \infty} x^{n!} = 1$. In practice, just two rules, where u is any word.

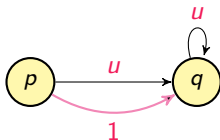


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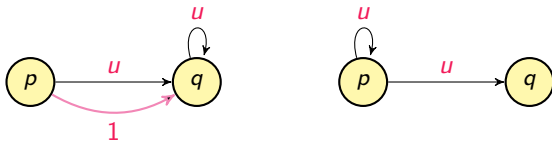


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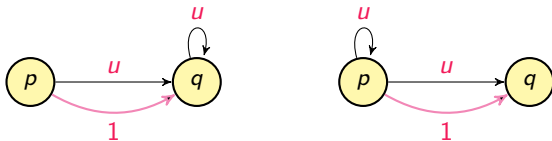


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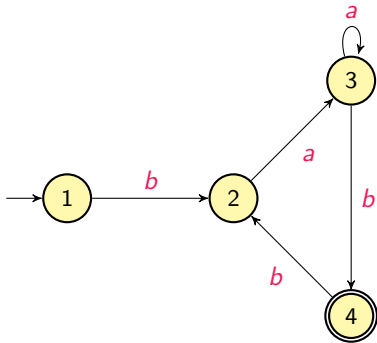
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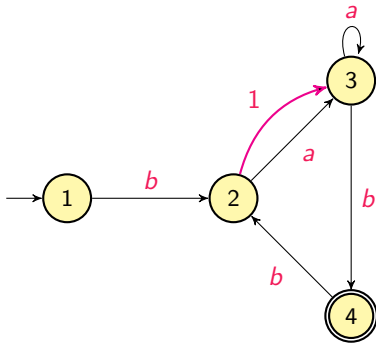
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An example: $L = (ba^+b)^+$ (borrowed from J.-É. Pin)

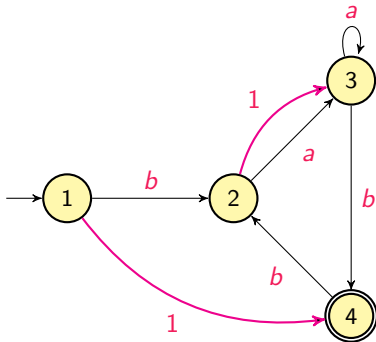


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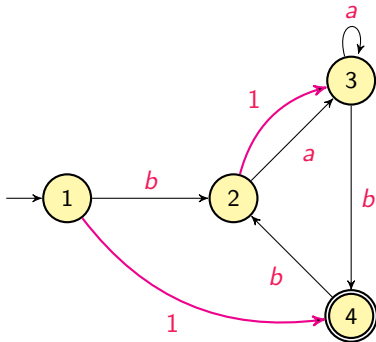
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Now, there are a path from 1 to 4 and a loop around 4, both labeled bb . The closure of L is $(ba^*b)^*$.

The Pin-Reutenauer algorithm for aperiodics

- ▶ To generalize this algorithm to \mathbf{A} (aperiodic semigroups), replace the free group by the algebra of terms built from X using multiplication and ω -power.
- ▶ Each such term can be interpreted on a profinite semigroup.
- ▶ $F_X^\omega \mathbf{A}$ is the subsemigroup of $\hat{F}_X \mathbf{A}$ generated by all such terms.
- ▶ $F_X^\omega \mathbf{A}$: ω -terms modulo equality on aperiodic semigroups.
- ▶ $X^+ \hookrightarrow F_X^\omega \mathbf{A} \hookrightarrow \hat{F}_X \mathbf{A}$.

Word problem for $F_X^\omega A$.

- ▶ Two distinct words can be separated by an aperiodic semigroup.
- ▶ But $(b(aa)^\omega)^{\omega+1}b$ and $b(a^\omega b)^\omega$ represent the same element of $F_X^\omega A$.

Theorem (McCammmond '01)

Using the following rewriting system, there is a procedure to transform any ω -word into a **normal form**: two ω -words are equal over $\hat{F}_X A$ if and only if they have the same normal form.

1. $(x^\omega)^\omega \longleftrightarrow x^\omega$
2. $(x^k)^\omega \longleftrightarrow x^\omega$ for $k \geq 2$
3. $x^\omega x^\omega \longleftrightarrow x^\omega$
4. $(xy)^\omega x \longleftrightarrow x(yx)^\omega$
5. $x^\omega x \longleftrightarrow x^\omega \longleftrightarrow xx^\omega$

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The **rank** of $w \in F_X^\omega A$ is the maximum nesting of ω -powers in the term in normal form representing w .

Notation: Closures for profinite topologies

$L \subseteq T$ topological semigroup: $\text{cl}_T(L)$ denotes the closure of L in T .

- ▶ We work in $\text{pv } A$, the pseudovariety of aperiodic semigroups.

$$\text{cl}(L) \stackrel{\text{def}}{=} \text{cl}_{\hat{F}_X A}(L) \qquad \text{cl}_\omega(L) \stackrel{\text{def}}{=} \text{cl}_{F_X^\omega A}(L)$$

- ▶ The topology on $F_X^\omega A$ is the induced topology in $\hat{F}_X A$:

$$\text{cl}_\omega(L) = \text{cl}(L) \cap F_X^\omega A.$$

- ▶ $\text{cl}_\omega(L)$ is **countable**. Algorithm to compute it?

Notation: algebraic closures

Let T be an ω -semigroup, and $L \subseteq T$.

$\langle L \rangle_\omega = \omega$ -subsemigroup of T generated by L .
(in practice in $L \subseteq F_X^\omega S$ or $L \subseteq F_A^\omega S$)

The Pin-Reutenauer algorithm for A

Theorem [Almeida, J.C. Costa, Z.]

The Pin-Reutenauer procedure for A: For all nonempty rational languages $K, L \subseteq X^+$:

$$\text{cl}_\omega(KL) = \text{cl}_\omega(K) \cdot \text{cl}_\omega(L), \quad (1)$$

$$\text{cl}_\omega(L^+) = \langle \text{cl}_\omega(L) \rangle_\omega. \quad (2)$$

Remarks

- ▶ Unfortunately, requires some more work to test whether an intersection of closures is empty.
- ▶ $\text{cl}_\omega(KL) \supseteq \text{cl}_\omega(K) \cdot \text{cl}_\omega(L)$ and $\text{cl}_\omega(L^+) \supseteq \langle \text{cl}_\omega(L) \rangle_\omega$ always hold (multiplication and ω -power are continuous).

The Pin-Reutenauer for A : Case of the product

- ▶ $\text{cl}_\omega(KL) \supseteq \text{cl}_\omega(K) \cdot \text{cl}_\omega(L)$ by continuity of multiplication.
- ▶ For the reverse implication, use the fact that A is ω -factorial.
Every factor in $\hat{F}_X A$ of an element of $F_X^\omega A$ is again in $F_X^\omega A$.

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 - ▶ By compactness, one can assume (x_n) and (y_n) convergent to $x \in \text{cl}(K)$ and $y \in \text{cl}(L)$.

The Pin-Reutenauer for A : Case of the product

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- ▶ For the reverse implication, use the fact that A is ω -factorial.
Every factor in $\hat{F}_X A$ of an element of $F_X^\omega A$ is again in $F_X^\omega A$.
- ▶ Proof sketch: take $w \in \text{cl}_\omega(KL)$.
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 - ▶ Since A is ω -factorial and $w = xy$, we get $x, y \in F_X^\omega A$.
 - ▶ So $x \in \text{cl}_\omega(K)$, and $y \in \text{cl}_\omega(L)$, whence $w \in \text{cl}_\omega(K) \cdot \text{cl}_\omega(L)$.

The case of the iteration

- ▶ The interesting inclusion is $\text{cl}_\omega(L^+) \subseteq \langle \text{cl}_\omega(L) \rangle_\omega$.
- ▶ We must prove that an element of $\text{cl}_\omega(L^+)$ can be written as a **finite term** as an ω -term, on elements of $\langle \text{cl}_\omega(L) \rangle_\omega$.
- ▶ Main problem: a sequence converging to w can have an unbounded number of factors.
- ▶ Two ingredients:
 1. Induction on the **rank** of w ,
 2. Aperiodic languages $L_n(w)$ giving a neighborhood basis of w in $F_X^\omega A$. $L_n(w)$ defined from the normal form of w by replacing all " ω " by " $> n$ ", with $L^{>n} = L^n L^+$.

Two natural questions

1. Automata for term languages.

- ▶ (Henckell's algorithm) Given regular $K, L \subseteq X^+$, one can decide whether

$$\text{cl}(K) \cap \text{cl}(L) = \emptyset.$$

- ▶ By a weak form of ω -reducibility for \mathbf{A} , this is equivalent

$$\text{cl}_\omega(K) \cap \text{cl}_\omega(L) = \emptyset.$$

Is it possible to test it using automata accepting languages in $F_X^\omega \mathbf{A}$?

2. For pseudovarieties enjoying a property called fullness, the PR algorithm is inherited by taking sub-pseudovarieties.

The pseudovariety \mathbf{S} has this property.

Does the Pin-Reutenauer algorithm hold for \mathbf{S} and signature $ab, a^{\omega-1}$?