

D1 : Topological foundations of the Stone-Priestley duality for recognizable languages (Task 5).

Cette partie de la tâche 5 est à peu près achevée, mais n'est pas encore publiée, car elle fera partie d'une publication plus importante. Le document en attaché qui résume ces travaux doit donc rester confidentiel.

Pervin spaces*

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Choosing the appropriate level of abstraction for the topological aspects of this article was a difficult decision. Metric spaces did not cover our needs and even uniform spaces did not suffice when dealing with lattices. Quasi-uniform spaces, on the other hand, while fulfilling our requirements, seemed to be too general a tool for our purpose. Our final decision was to briefly recall the basic definitions and main properties of quasi-uniformities and then to switch to a very special case, the Pervin spaces. To our surprise, turning to Pervin spaces did not only simplify a number of results and proofs, but also lead us to an unconventional point of view on Stone’s duality.

Stone’s representation theorem characterizes bounded distributive lattices as the lattices of compact open sets of certain topological spaces. Here we consider a lattice of subsets of a fixed set X and we attach to this lattice a quasi-uniform structure, turning X into a Pervin space. It turns out that the completion of this quasi-uniform space is a compact space, which is precisely the Stone dual of the original lattice. This fact was certainly known to topologists (at least in the Boolean case), but was apparently never put forwards, probably for the following reason. A major consequence of Stone’s theorem is that any bounded distributive lattice is isomorphic to a lattice of sets, so that proving Stone’s theorem only for lattices of sets may appear at first sight as a regression. There is, however, some interest in this approach (TO DO).

Even in the simplified setting of Pervin spaces, a certain familiarity with general topology is certainly useful to understand this paper and we hope this is not too demanding for the reader. On the other hand, this is not the unique area in computer science where topology is mandatory, and specialists of domain theory are even more demanding in this respect.

1 Relations, preorders and filters

1.1 Relations

Let X be a set. The subsets of $X \times X$ can be viewed as relations on X . In particular, if U and V are subsets of $X \times X$, we use the notation UV to denote the composition of the two relations, that is, the set

$$UV = \{(x, y) \in X \times X \mid \text{there exists } z \in X, (x, z) \in U \text{ and } (z, y) \in V\}.$$

Given a relation U , the *inverse relation* of U is the relation

$$U^{-1} = \{(x, y) \in X \times X \mid (y, x) \in U\}$$

and the *symmetrized relation* of U is the relation

$$U^s = U \cap U^{-1}$$

A relation U is *symmetrical* if $U = U^{-1}$. Finally, if $x \in X$ and $U \subseteq X \times X$, we write $U(x)$ for the set $\{y \in X \mid (x, y) \in U\}$.

A *preorder* is a reflexive and transitive relation. Let \leq be a preorder on a set X . A subset S of X is an *upper set* if the conditions $x \in S$ and $x \leq y$ imply $y \in S$. It is a *lower set* if the conditions $x \in S$ and $y \leq x$ imply $y \in S$.

1.2 Filters

Filters can be defined on a set, or more generally one can define filters of a partially ordered set. We choose a middle way, by defining filters of a lattice of subsets of a given set.

Let X be a set. A *lattice of subsets* of X is a set of subsets of X containing \emptyset (the *empty set*) and X (the *full set*) and which is closed under finite union and finite intersection. A *Boolean algebra* is a lattice closed under complement. In particular, $\mathcal{P}(X)$, the set of all subsets of X , is a Boolean algebra.

Let X be a set and let \mathcal{L} be a lattice of subsets of X . A *filter* of \mathcal{L} is a subset \mathcal{F} of \mathcal{L} such that:

- (F₁) The empty set does not belong to \mathcal{F} , but the full set belongs to \mathcal{F} ,
- (F₂) If $F \in \mathcal{F}$, $G \in \mathcal{L}$ and $F \subseteq G$, then $G \in \mathcal{F}$ (closure under extension),
- (F₃) If $F, G \in \mathcal{F}$, then $F \cap G \in \mathcal{F}$ (closure under intersection).

A *filter basis* is a subset \mathcal{F} of \mathcal{L} satisfying conditions (F₁) and (F₃). The filter generated by a filter basis \mathcal{B} is the set of elements of \mathcal{L} containing an element of \mathcal{B} . In particular, \mathcal{B} is a basis for a filter \mathcal{F} if and only if every set of \mathcal{F} contains a set of \mathcal{B} .

A filter is *prime* if it satisfies the further condition

- (F₄) If F and G are elements of \mathcal{L} such that $F \cup G \in \mathcal{F}$, then $F \in \mathcal{F}$ or $G \in \mathcal{F}$.

Filters are naturally ordered by inclusion. A filter is said to be *maximal* if it is maximal for this order.

The notion of an ultrafilter only makes sense in a Boolean algebra: a filter \mathcal{F} on a Boolean algebra \mathcal{B} is an *ultrafilter* if for each $L \in \mathcal{B}$, either $L \in \mathcal{F}$ or $L^c \in \mathcal{F}$. The next proposition summarize the connections between these three notions.

Proposition 1.1 *In a lattice of subsets, every maximal filter is prime but the converse is not true in general. In a Boolean algebra, the notions of prime filter, maximal filter and ultrafilter are equivalent.*

Let \mathcal{L} be a lattice of subsets of X and let \mathcal{B} be the Boolean algebra generated by \mathcal{L} . There is a nice connection between the prime filters of \mathcal{L} and those of \mathcal{B} , discovered by Nerode [7, p. 398]. Given a prime filter \mathcal{F} of \mathcal{L} let \mathcal{F}^\sharp be the filter of \mathcal{B} generated by $\mathcal{F} \cup \{L^c \mid L \in \mathcal{L} - \mathcal{F}\}$.

Proposition 1.2 (Nerode's trick) *The maps $\mathcal{F} \rightarrow \mathcal{F}^\sharp$ and $\mathcal{G} \rightarrow \mathcal{G} \cap \mathcal{L}$ define mutually inverse bijections between the set of prime filters of \mathcal{L} and the set of prime filters of \mathcal{B} .*

A filter of \mathcal{L} is *principal* if it is generated by a single element, that is, if it is of the form

$$\uparrow L = \{F \in \mathcal{L} \mid L \subseteq F\}$$

for some element $L \in \mathcal{L}$. If L is the singleton $\{x\}$, we simplify the notation $\uparrow\{x\}$ to $\uparrow x$, so that

$$\uparrow x = \{F \in \mathcal{L} \mid x \in F\}$$

Note that $\uparrow x$ is a prime filter, since if $x \in F \cup G$, then either $x \in F$ or $x \in G$.

Let \mathcal{L} be lattice of subsets of X . An *ideal* of \mathcal{L} is a nonempty lower subset of \mathcal{L} closed under finite union. We will need a weak version of a result known as the *prime filter theorem*.

Theorem 1.3 *Let \mathcal{I} be an ideal of \mathcal{L} and let L be an element of $\mathcal{L} - \mathcal{I}$. Then there is a prime filter \mathcal{F} containing L and disjoint from \mathcal{I} .*

Proposition 1.4 (Krull) *If \mathcal{F} is a filter of \mathcal{L} and $L \notin \mathcal{F}$, then there is a prime filter \mathcal{P} containing \mathcal{F} with $L \notin \mathcal{P}$.*

1.3 Prime spectrum of a lattice of subsets

Let X be a set and let \mathcal{L} be a lattice of subsets of X . The *prime spectrum* of \mathcal{L} is the set of its prime filters, ordered by inclusion. There is another convenient description of the prime spectrum making use of valuations. A *valuation* on \mathcal{L} is a lattice morphism from \mathcal{L} into the Boolean lattice $\{0, 1\}$. In other words, a valuation is a function v from \mathcal{L} into $\{0, 1\}$ satisfying the following properties: $v(\emptyset) = 0$, $v(X) = 1$ and for all $L, L' \in \mathcal{L}$, $v(L \cup L') = v(L) + v(L')$ and $v(L \cap L') = v(L)v(L')$, where the addition and the product denote the Boolean operations. Valuations are naturally ordered by setting $v \leq v'$ if and only if $v(L) \leq v'(L)$ for all $L \in \mathcal{L}$.

The connection between valuations and prime filters is now easy to state. To each prime filter \mathcal{F} , let us attach a function $v_{\mathcal{F}} : \mathcal{L} \rightarrow \{0, 1\}$ by setting $v_{\mathcal{F}}(L) = 1$ if and only if $L \in \mathcal{F}$. We also attach to each valuation v a filter \mathcal{F}_v defined by $\mathcal{F}_v = \{L \in \mathcal{L} \mid v(L) = 1\}$.

Proposition 1.5 *The maps $\mathcal{F} \mapsto v_{\mathcal{F}}$ and $v \mapsto \mathcal{F}_v$ are mutually inverse order-preserving bijections between the set of prime filters of \mathcal{L} and the set of valuations on \mathcal{L} .*

TO DO: Donne deux façons de définir le spectre + la topologie vient plus tard.

2 Topology

In this section, we recall the notions of topology needed to read this paper.

2.1 Topological spaces

Recall that a *topology* on a set X is a set \mathcal{T} of subsets of X satisfying the following conditions:

- (T₁) The empty set and X are in \mathcal{T} ,
- (T₂) The union of arbitrarily many elements of \mathcal{T} is an element of \mathcal{T} ,
- (T₃) The intersection of finitely many elements of \mathcal{T} is an element of \mathcal{T} .

The elements of \mathcal{T} are called the *open sets*. The complement of an open set is called a *closed set*. A set is *clopen* if it is both open and closed. A *neighbourhood* of a point x of X is a set which contains an open set containing x . A *topological space* is a set together with a topology on it.

It is sometimes convenient to give a *basis* for a topology on X . This is a collection \mathcal{B} of open subsets of X such that every open set is the union of elements of \mathcal{B} . An equivalent condition is that \mathcal{B} satisfies the two following condition:

(B₁) every finite intersection of elements of \mathcal{B} is a union of elements of \mathcal{B} .

The open sets of the topology generated by \mathcal{B} are by definition the arbitrary unions of elements of \mathcal{B} .

The *closure* of a subset S of X , denoted by \overline{S} , is the intersection of the closed sets containing S . A subset of X is *dense* if its closure is equal to X . If S is a dense subset of X and O is an open set of X , then $S \cap O$ is dense in O .

If S is a subset of a topological space (X, \mathcal{T}) , then the traces $S \cap O$ for $O \in \mathcal{T}$, define a topology on S , called the *relative topology*.

A map from a topological space into another one is *continuous* if the inverse image of each open set is an open set. It is a *homeomorphism* if it is a continuous bijection and the inverse bijection is also continuous. Two topological spaces are *homeomorphic* if there is a homeomorphism between them.

2.2 Filters on a topological space

If X is a space, a filter of $\mathcal{P}(X)$ is simply called a *filter on X* . Similarly a filter basis for a filter of $\mathcal{P}(X)$ is called a *filter basis on X* .

Let X be a topological space. A point x of X is a *cluster point* of a filter basis \mathcal{B} if each neighbourhood of x meets every element of \mathcal{B} . A filter basis \mathcal{B} *converges* to a point x of X if, for each neighbourhood U of x , there is a set B of \mathcal{B} such that $B \subseteq U$. In this case, x is called a *limit* of \mathcal{B} and \mathcal{B} is a *convergent* filter basis.

2.3 Separation properties

Let X be a topological space. Let us denote by $\mathcal{N}(x)$ the filter of neighbourhoods of a point x . Two points x and y of X are *topologically distinguishable* if they do not have exactly the same neighbourhoods: $\mathcal{N}(x) \neq \mathcal{N}(y)$. This is the case if and only if there exists a neighbourhood of one of them which does not contain the other or, equivalently, if the closures of x and y are distinct sets. A space is *Kolmogorov* (or T_0) if any two distinct points are topologically distinguishable.

Two points x and y of X are *separated* if each of them has a neighbourhood that is not a neighbourhood of the other: $\mathcal{N}(x) \not\subseteq \mathcal{N}(y)$ and $\mathcal{N}(y) \not\subseteq \mathcal{N}(x)$. This is the case if and only if there exists a neighbourhood of x which does not contain y and a neighbourhood of y which does not contain x or, equivalently, if the closures of x and y are disjoint sets. A space is *accessible* (or T_1) if any two distinct points are separated or, equivalently, if each singleton set is closed.

Two points x and y of X are *separated by neighbourhoods* if they have disjoint neighbourhoods. A space is *Hausdorff* (or T_2) if any two distinct points are separated by neighbourhoods.

Variante

A topological space X is a *Kolmogorov space* (or T_0 -space) if for any two distinct points of X , there is an open set which contains one of these points and not the other. It is an *accessible space* (or T_1 -space) if for any two distinct points x and y in X , there is an open set which contains x but not y or, equivalently, if each singleton set is closed. It is a *Hausdorff space* (or T_2 -space) if any two distinct points of X have disjoint neighbourhoods.

A compléter: A space is *totally disconnected* if its connected components are the one-point sets.

2.4 Specialization preorder

The *specialization preorder* on a topological space X is the relation \leq defined on X by $x \leq y$ if and only if $\overline{\{x\}} \subseteq \overline{\{y\}}$ or, equivalently, if and only if $x \in \overline{\{y\}}$. Equivalently $x \leq y$ if and only if any neighbourhood of x is a neighbourhood of y or if and only if y belongs to every open set that contains x . Consequently, every open set is an upper set with respect to \leq and every closed set is a lower set. A continuous function between two topological spaces is monotone with respect to the specialization preorders of these spaces.

The specialization preorder is an order if and only if X is Kolmogorov and it is the equality if and only if X is accessible.

2.5 Compact spaces

An *open cover* of a topological space is a family of open sets whose union is the whole space. A topological space is *compact* if every open covers have a finite subcover.

3 Uniform and quasi-uniform spaces

This section surveys the basic definitions and results on uniform spaces that will be needed in the sequel. Our basic references are [1, 2] for uniform spaces and [4, 6] for quasi-uniform spaces.

3.1 Definitions

A *quasi-uniformity* on a set X is a non empty set \mathcal{U} of subsets of $X \times X$ satisfying the following properties:

- (U₁) if a subset U of $X \times X$ contains an element of \mathcal{U} , then $U \in \mathcal{U}$,
- (U₂) the intersection of any two elements of \mathcal{U} contains an element of \mathcal{U} ,
- (U₃) each element of \mathcal{U} contains the diagonal of $X \times X$,
- (U₄) for each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $VV \subseteq U$.

A quasi-uniformity is called a *uniformity* if it satisfies the additional condition

- (U₅) for each $U \in \mathcal{U}$, $U^{-1} \in \mathcal{U}$.

The elements of a quasi-uniformity are called *entourages*. Note that $X \times X$ is always an entourage. The pair (X, \mathcal{U}) (or the set X if \mathcal{U} is understood) is called a *quasi-uniform space*. It is a *uniform space* if \mathcal{U} is a uniformity. The *discrete uniformity* on X is the unique uniformity which contains the diagonal of $X \times X$.

A *basis* of a quasi-uniformity [uniformity] \mathcal{U} is a subset \mathcal{B} of \mathcal{U} such that each element of \mathcal{U} contains an element of \mathcal{B} . In particular, \mathcal{U} consists of all the relations on X containing an element of \mathcal{B} . We say that \mathcal{U} is *generated* by \mathcal{B} . A set \mathcal{B} of subsets of $X \times X$ is a basis of some quasi-uniformity [uniformity] if and only if it satisfies properties (U₂) to (U₄) [(U₂) to (U₅)].

A *subbasis* of a quasi-uniformity \mathcal{U} is a subset \mathcal{B} of \mathcal{U} such that the finite intersections of members of \mathcal{B} form a basis of \mathcal{U} . A set of subsets of $X \times X$ is a subbasis of some quasi-uniformity if and only if it satisfies properties (U₃) and (U₄).

A basis of a quasi-uniformity is *symmetrical* if all its elements are symmetrical. The quasi-uniformity generated by a symmetrical basis is actually a uniformity. If \mathcal{B} is a basis of a quasi-uniformity \mathcal{U} , the entourages U^s , for $U \in \mathcal{B}$, form a symmetrical basis of a uniformity \mathcal{U}^s , which is the smallest uniformity containing \mathcal{U} . In particular \mathcal{U}^s is also generated by the entourages U^s , for $U \in \mathcal{U}$.

If X and Y are quasi-uniform spaces, a function $\varphi: X \rightarrow Y$ is said to be *uniformly continuous* if, for each entourage V of Y , $(\varphi \times \varphi)^{-1}(V)$ is an entourage of X , or, equivalently, if for each entourage V of Y , there exists an entourage U of X such that $(\varphi \times \varphi)(U) \subseteq V$.

Proposition 3.1 *Let X and Y be two quasi-uniform spaces and let \mathcal{B} be a subbasis of the quasi-uniformity of Y . Then a function φ from X to Y is uniformly continuous if and only if, for every element B of \mathcal{B} , $(\varphi \times \varphi)^{-1}(B)$ is an entourage of X .*

Proof. Let V be an entourage of Y . Since \mathcal{B} is a subbasis, V contains an entourage B which is the intersection of a finite family $(B_i)_{i \in F}$ of elements of \mathcal{B} . For each i in F , the set $(\varphi \times \varphi)^{-1}(B_i)$ is by hypothesis an entourage of X . Since

$$(\varphi \times \varphi)^{-1}(B) = (\varphi \times \varphi)^{-1}\left(\bigcap_{i \in F} B_i\right) = \bigcap_{i \in F} (\varphi \times \varphi)^{-1}(B_i)$$

the set $(\varphi \times \varphi)^{-1}(B)$ is an entourage of X contained in $(\varphi \times \varphi)^{-1}(V)$. It follows that $(\varphi \times \varphi)^{-1}(V)$ itself is an entourage of X . Therefore, φ is uniformly continuous. \square

An *isomorphism* between two quasi-uniform spaces X and Y is a bijection from X onto Y such that both φ and φ^{-1} are uniformly continuous.

Let $(X_i, \mathcal{U}_i)_{i \in I}$ a family of quasi-uniform spaces and let $X = \prod_{i \in I} X_i$. The *product quasi-uniformity* on X is the coarsest quasi-uniformity for which all the projections $\pi_i : X \rightarrow X_i$ are uniformly continuous. A subbase for this quasi-uniformity consists of all the sets of the form $\{(x, y) \mid (\pi_i(x), \pi_i(y)) \in U\}$ for $i \in I$ and $U \in \mathcal{U}_i$. The quasi-uniform space (X, \mathcal{U}) is called the product of the family $(X_i, \mathcal{U}_i)_{i \in I}$.

A quasi-uniformity is said to be *transitive* if it has a basis consisting of transitive entourages. Transitivity is a quasi-uniform invariant. More precisely, if (X, \mathcal{U}) and (Y, \mathcal{V}) are isomorphic quasi-uniform spaces and if \mathcal{U} is transitive, then \mathcal{V} is transitive.

3.2 Induced topology

Let \mathcal{U} be a quasi-uniformity on X . For each $x \in X$, let $\mathcal{U}(x) = \{U(x) \mid U \in \mathcal{U}\}$. There exists a unique topology on X , called the *topology induced* by \mathcal{U} , for which $\mathcal{U}(x)$ is the set of neighbourhoods of x for each $x \in X$. It follows that a subset S of X is open if and only if, for every $x \in S$, there is an entourage U of X such that $U(x) \subseteq S$.

Proposition 3.2 ([6, Theorem 1.24], [4, Proposition 1.14]) *A uniformly continuous function is continuous for the topology induced by \mathcal{U} and for the topology induced by \mathcal{U}^s .*

Let \mathcal{U} be a [quasi-]uniformity on X . The intersection of all the entourages of \mathcal{U} is an equivalence relation [a preorder] on X , called the *uniform equivalence* [preorder] associated with \mathcal{U} and denoted by $\leq [\sim]$.

Proposition 3.3 *A uniformly continuous function between two quasi-uniform spaces is monotone with respect to the uniform preorders of these spaces.*

Proof. Let φ be a uniformly continuous map from X into Y . Let $s, t \in X$ with $s \leq t$. By definition, one has $(s, t) \in U$ for all entourages U of X . If V is an entourage of Y , then $(\varphi \times \varphi)^{-1}(V)$ is an entourage of X . Therefore $(s, t) \in (\varphi \times \varphi)^{-1}(V)$ and $(\varphi(s), \varphi(t)) \in V$. It follows that $\varphi(s) \leq \varphi(t)$ and thus φ is monotone. \square

A quasi-uniform space is *Hausdorff* [Kolmogorov] if its induced topology is Hausdorff [Kolmogorov].

Proposition 3.4 ([4, Proposition 1.9]) *Let X be a quasi-uniform space.*

- (1) *It is Kolmogorov if and only if the intersection of all its entourages is a partial order.*
- (2) *It is accessible if and only if the intersection of all its entourages is equal to the diagonal of $X \times X$.*
- (3) *It is Hausdorff if and only if the intersection of all the sets of the form UU^{-1} , where U is an entourage, is equal to the diagonal of $X \times X$.*

3.3 Kolmogorov and Hausdorff quotients

Let (X, \mathcal{U}) be a quasi-uniform space. The intersection of all the entourages of \mathcal{U}^s is an equivalence relation \sim on X . Let $\pi : X \rightarrow X/\sim$ be the natural map.

Proposition 3.5 *Let (X, \mathcal{U}) be a quasi-uniform space. The sets of the form $(\pi \times \pi)(U)$ where U is an entourage of X form a subbasis of a quasi-uniformity on X/\sim . The quasi-uniform space X/\sim is Kolmogorov and the map $\pi : X \rightarrow X/\sim$ is uniformly continuous.*

Proof. For the first part of the statement, the only nontrivial part is to prove that condition (U_4) is satisfied.

Let U be an entourage of X . An iterative application of condition (U_4) shows that there exists an entourage V such that $V^6 \subseteq U$. Let $T = (\pi \times \pi)(V)$. We claim that TT is a subset of $(\pi \times \pi)(U)$. Indeed, let x and y be two points

of X such that $(\pi(x), \pi(y)) \in TT$. Then there exists a point z of X such that $(\pi(x), \pi(z)) \in T$ and $(\pi(z), \pi(y)) \in T$. Since $T = (\pi \times \pi)(V)$, there exist some points x', z', z'' and y' such that $(x', z') \in V$, $(z'', y') \in V$, $\pi(x) = \pi(x')$, $\pi(y) = \pi(y')$ and $\pi(z) = \pi(z') = \pi(z'')$. Then by definition of π , one gets $(x, x') \in V$, $(x', z) \in V$, $(z, z'') \in V$ and $(y', y) \in V$. It follows that $(x, y) \in V^6$ and hence $(x, y) \in U$. Therefore $(\pi(x), \pi(y)) \in (\pi \times \pi)(U)$, which proves the claim and the first part of the statement.

A similar argument can be used to show that X/\sim is Kolmogorov. Let x and y be two points of X such that $\pi(x) \neq \pi(y)$. Then there is an entourage U of \mathcal{U} such that $(x, y) \notin U^s$. Since $U^s = U \cap U^{-1}$, one gets $(x, y) \notin U$ or $(y, x) \notin U$. Suppose that $(x, y) \notin U$ (the other case is symmetrical) and let v be an entourage such that $V^3 \subseteq U$. Let $T = (\pi \times \pi)(V)$. We claim that $(\pi(x), \pi(y)) \notin T$. Indeed, if $(\pi(x), \pi(y)) \in T$, then there exists $(x', y') \in V$ such that $\pi(x) = \pi(x')$ and $\pi(y) = \pi(y')$. It follows that both (x, x') and (y, y') belong to $V \cap V^{-1}$. Therefore $(x, x'), (x', y'), (y', x') \in V$, whence $(x, y) \in V^3$ and finally $(x, y) \in U$, a contradiction. This proves the claim and shows that the open set $T(\pi(x))$ does not contain $\pi(y)$. Thus X/\sim is Kolmogorov.

Finally, it is trivial to see that π is uniformly continuous. Indeed, for each entourage U , the set $(\pi \times \pi)^{-1}((\pi \times \pi)(U))$ contains U and is therefore an entourage. \square

Let us say that a quasi-uniform space Y is a *quotient* of a quasi-uniform space X if there exists a surjective uniformly continuous map from X onto Y .

The uniform space X/\sim defined by Proposition 3.5 is called the *Kolmogorov quotient* of X . When X is a uniform space, this quotient is called the *Hausdorff quotient* of X , in view of the following result:

Proposition 3.6 *Let (X, \mathcal{U}) be a uniform space. The sets of the form $(\pi \times \pi)(U)$ where U is an entourage of X form a subbasis of a uniformity on X/\sim . The uniform space X/\sim is Hausdorff and the map $\pi : X \rightarrow X/\sim$ is uniformly continuous.*

The Kolmogorov quotient satisfies the following universal property.

Proposition 3.7 *Let φ be a uniformly continuous map from a quasi-uniform space X to a Kolmogorov quasi-uniform space Y . Then there is a unique uniformly continuous map $\tilde{\varphi}$ from X/\sim to Y such that $\varphi = \tilde{\varphi} \circ \pi$.*

Proof. We first prove that if $x \sim y$, then $\varphi(x) = \varphi(y)$. Indeed, let V be an entourage of Y and let $U = (\varphi \times \varphi)^{-1}(V)$. Since φ is uniformly continuous, U is an entourage of X . If $x \sim y$, then (x, y) and (y, x) are both in U , which implies that $(\varphi(x), \varphi(y))$ and $(\varphi(y), \varphi(x))$ are both in V . Since this holds for any entourage V of Y , one has $\varphi(x) \sim \varphi(y)$ and thus $\varphi(x) = \varphi(y)$ since Y is Kolmogorov. Therefore, there is a unique map $\tilde{\varphi} : X/\sim \rightarrow Y$ such that $\varphi = \tilde{\varphi} \circ \pi$. It remains to show that $\tilde{\varphi}$ is uniformly continuous, which is easy. Indeed, with the notation above, one has $(\tilde{\varphi} \times \tilde{\varphi})^{-1}(V) = (\varphi \times \varphi)(U)$, which is by definition an entourage of X/\sim . \square

There is of course an analogous result for the Hausdorff quotient.

Proposition 3.8 *Let φ be a uniformly continuous map from a uniform space X to an Hausdorff uniform space Y . Then there is a unique uniformly continuous map $\tilde{\varphi}$ from X/\sim to Y such that $\varphi = \tilde{\varphi} \circ \pi$.*

4 Completeness

Let X be a quasi-uniform space. A filter \mathcal{F} on X is a *Cauchy filter* if, for every entourage U , there exists an $x \in X$ such that $U(x) \in \mathcal{F}$. One can show that if X is a uniform space, then \mathcal{F} is a Cauchy filter if and only if, for every entourage U , there exists an F in \mathcal{F} such that $F \times F \subseteq U$.

A minimal element (for the inclusion order) of the set of Cauchy filters is called a *minimal Cauchy filter*. The following result will serve as an example.

Proposition 4.1 *Let X be a quasi-uniform space. Then for each $x \in X$, the set of neighbourhoods of x is a minimal Cauchy filter.*

Proof. Let $x \in X$ and let U be an entourage. By definition of the topology, $U(x)$ is a neighbourhood of x and thus $\mathcal{N}(x)$ is a Cauchy filter. Let \mathcal{F} be a Cauchy filter contained in $\mathcal{N}(x)$. For every entourage U , there exists an $x \in X$ such that $U(x) \in \mathcal{F}$. It follows that $\mathcal{N}(x)$ is contained in \mathcal{F} . Therefore $\mathcal{N}(x)$ is a minimal Cauchy filter. \square

A uniform space is *complete* if every Cauchy filter is converging. The corresponding definition for quasi-uniform spaces is apparently weaker: A quasi-uniform space is *complete* if every Cauchy filter has a cluster point. Fortunately, the two definitions are coherent since in a uniform structure, a Cauchy filter converges to each of its cluster points.

Question (dans le cas général et dans le cas Pervin): Si (X, \mathcal{U}) et (X, \mathcal{U}^{-1}) sont complets, est-ce que (X, \mathcal{U}^s) est complet (et réciproquement)?

A quasi-uniform space is *totally bounded* if, for each entourage U of \mathcal{B} , there exists a finite cover \mathcal{C} of X such that $C \times C \subseteq U$ for each $C \in \mathcal{C}$. Totally bounded quasi-uniform spaces are closed under quotients.

Proposition 4.2 *A quotient of a totally bounded quasi-uniform space is totally bounded.*

Proof. Let X and Y be two totally bounded quasi-uniform spaces and let φ be a surjective uniformly continuous map from X onto Y . Let V be an entourage of Y and let $U = (\varphi \times \varphi)^{-1}(V)$. Then U is an entourage of X and there exists a finite cover \mathcal{C} of X such that $C \times C \subseteq U$ for each $C \in \mathcal{C}$. Now $\varphi(C)$ is a cover of Y and for each $C \in \mathcal{C}$, one has $\varphi(C) \times \varphi(C) \subseteq (\varphi \times \varphi)(U) \subseteq V$. This shows that Y is totally bounded. \square

The interest of totally bounded quasi-uniformities lies in the following result [6, p. 52].

Proposition 4.3 *A quasi-uniform space is compact if and only if it is complete and totally bounded.*

4.1 Completion of a uniform space

Theorem 4.4 *Let X be a uniform space. There exists a complete Hausdorff uniform space \widehat{X} and a uniformly continuous mapping $\iota: X \rightarrow \widehat{X}$ having the following universal property: for each uniformly continuous mapping φ from X into a complete Hausdorff uniform space Y , there exists a unique uniformly continuous mapping $\widehat{\varphi}: \widehat{X} \rightarrow Y$ such that $\widehat{\varphi} \circ \iota = \varphi$.*

The complete Hausdorff uniform space \widehat{X} given by Theorem 4.4 is called the *Hausdorff completion* of X and the application $\iota: X \rightarrow \widehat{X}$ is the *canonical mapping* from X into its Hausdorff completion. It can be constructed as follows (see [2]). Take for \widehat{X} the set of minimal Cauchy filters on X . For each symmetric entourage V of \mathcal{U} , consider the set \widetilde{V} of all pairs $(\mathcal{F}_1, \mathcal{F}_2)$ of elements of \widehat{X} such that V belongs to $\mathcal{F}_1 \cap \mathcal{F}_2$. The sets \widetilde{V} form the basis of a uniformity on \widehat{X} .

4.2 Completion of a quasi-uniform space

La suite provient de LindgrenFletcher78, Romaguera 2000 et Kunzi 2000.

A *completion* of X is a complete quasi-uniform space (Y, \mathcal{V}) such that (X, \mathcal{U}) is quasi-isomorphic to a \mathcal{V}^s -dense subset of Y . It was proved in [5, 4] that every quasi-uniform space has an a completion. Moreover, (Y, \mathcal{V}^{-1}) is a completion of (X, \mathcal{U}^{-1}) . In addition, a Kolmogorov quasi-uniform space (X, \mathcal{U}) has a (up to quasi-uniform isomorphism) unique Kolmogorov completion $(\widehat{X}, \widehat{\mathcal{U}})$. The uniformities $(\widehat{\mathcal{U}})^s$ and $\widehat{\mathcal{U}}^s$ coincide. Furthermore, if φ is a quasi-uniformly continuous map from (X, \mathcal{U}) into a complete quasi-uniform Kolmogorov space (Y, \mathcal{V}) , then φ admits a (unique) uniformly continuous extension $\widehat{\varphi}: \widehat{X} \rightarrow Y$. Let us detail this construction.

Let (X, \mathcal{U}) be a quasi-uniform space. If \mathcal{F}, \mathcal{G} and $\mathcal{F} \vee \mathcal{G}$ are filter bases on X , then $(\mathcal{F}, \mathcal{G})$ is called a *pair filter basis*. An element x of X is a *cluster point* of $(\mathcal{F}, \mathcal{G})$ if it is a cluster point of \mathcal{F} in the topology induced by \mathcal{U} and a cluster point of \mathcal{G} in the topology induced by \mathcal{U}^{-1} .

For each $x \in X$, let $L(x) [R(x)]$ be the filter generated by $\{U^{-1}(x) \mid U \in \mathcal{U}\}, \{\{U(x) \mid U \in \mathcal{U}\}\}$. Then $(L(x), R(x))$ is a pair filter basis.

A filter basis \mathcal{F}' is *coarser* than a filter basis \mathcal{F} if the filter generated by \mathcal{F}' is contained in the filter generated by \mathcal{F} . A pair filter basis $(\mathcal{F}', \mathcal{G}')$ is *coarser* than a pair filter basis $(\mathcal{F}, \mathcal{G})$ if \mathcal{F}' is coarser than \mathcal{F} and \mathcal{G}' is coarser than \mathcal{G} . If $(\mathcal{F}', \mathcal{G}')$ is coarser than $(\mathcal{F}, \mathcal{G})$ and $(\mathcal{F}, \mathcal{G})$ is coarser than $(\mathcal{F}', \mathcal{G}')$, then $(\mathcal{F}', \mathcal{G}')$ and $(\mathcal{F}, \mathcal{G})$ are said to be *equivalent*.

A pair filter $(\mathcal{F}, \mathcal{G})$ *converges* to x provided that $(L(x), R(x))$ is coarser than $(\mathcal{F}, \mathcal{G})$. A pair filter basis $(\mathcal{F}, \mathcal{G})$ is *Cauchy* provided that for each $U \in \mathcal{U}$, there is an $F \in \mathcal{F}$ and a $G \in \mathcal{G}$ such that $F \times G \subseteq U$. A Cauchy pair filter basis is *minimal* if any Cauchy pair filter basis coarser than it is equivalent to it.

If (X, \mathcal{U}) is a uniform space and \mathcal{F} is a filter on X then clearly $\mathcal{U} = \mathcal{U}^s$ and \mathcal{F} is a Cauchy [convergent] filter on X if, and only if, $(\mathcal{F}, \mathcal{F})$ is a Cauchy [convergent] pair filter basis.

The following results are proved in [5].

Proposition 4.5 *Every convergent pair filter basis is a Cauchy pair filter basis and for each $x \in X$, $(L(x), R(x))$ is a minimal Cauchy pair filter basis.*

Let $(\mathcal{F}, \mathcal{G})$ be a Cauchy pair filter basis on X .

- (1) If x is a cluster point of $(\mathcal{F}, \mathcal{G})$, then $(\mathcal{F}, \mathcal{G})$ converges to x .
- (2) \mathcal{F} has a basis that is open in the topology induced by \mathcal{U}^{-1} and \mathcal{G} has a basis that is open in the topology induced by \mathcal{U} .
- (3) There is exactly one minimal Cauchy pair filter $(\mathcal{F}_0, \mathcal{G}_0)$ that is coarser than $(\mathcal{F}, \mathcal{G})$. A basis for \mathcal{F}_0 is $\{U^{-1}(F) \mid U \in \mathcal{U} \text{ and } F \in \mathcal{F}\}$ and a basis for \mathcal{G}_0 is $\{U(G) \mid U \in \mathcal{U} \text{ and } G \in \mathcal{G}\}$.
- (4) Let Y be a quasi-uniform space and let $\varphi : X \rightarrow Y$ be a uniformly continuous function. Then $(\varphi(\mathcal{F}), \varphi(\mathcal{G}))$ is a Cauchy pair filter basis on Y .

4.2.1 Kolmogorov case

We assume now that (X, \mathcal{U}) is a Kolmogorov quasi-uniform space, or, equivalently, that (X, \mathcal{U}^s) is Hausdorff. We say that X is *complete* provided that each Cauchy pair filter basis converges. It is equivalent to require that (X, \mathcal{U}^s) is a complete uniform space. Standard results on uniform spaces can be extended to this case. See [4, p. 50, Theorem 3.10 and p. 51, Theorem 3.11] and [6, p. 48, Theorem 4.7].

Proposition 4.6

- (1) A subspace of a complete quasi-uniform space (X, \mathcal{U}) is complete, if, and only if it is a closed set in the topology induced by \mathcal{U}^s .
- (2) The product of any family of quasi-uniform spaces is complete if, and only if, each factor space is complete.

A *completion* of X is a complete quasi-uniform space Y that has a dense subspace isomorphic to X .

Theorem 4.7 *Every Kolmogorov quasi-uniform space has a completion.*

The construction of this completion is similar to the one of a uniform space. Let \widehat{X} be the set of all minimal \mathcal{U}^s -Cauchy filters on X . For each $U \in \mathcal{U}$, let

$$\widehat{U} = \{(\mathcal{F}, \mathcal{G}) \mid \text{there exists } F \in \mathcal{F} \text{ and } G \in \mathcal{G} \text{ such that } F \times G \subseteq U\}$$

Then the set $\{\widehat{U} \mid U \in \mathcal{U}\}$ is a basis for a quasi-uniformity $\widehat{\mathcal{U}}$ on \widehat{X} . Further, the map from X into \widehat{X} which maps each point x to the filter $\mathcal{N}(x)$ of neighbourhoods of x (for the topology induced by \mathcal{U}^s) is a uniformly continuous embedding.

The completion enjoys the following universal property:

Theorem 4.8 *Let (X, \mathcal{U}) be a Kolmogorov quasi-uniform space, let (Y, \mathcal{V}) be a Kolmogorov complete quasi-uniform space and let $\varphi : X \rightarrow Y$ be a uniformly continuous function. Then φ admits a unique $\widehat{\mathcal{U}}^s$ - \mathcal{V}^s continuous extension $\widehat{\varphi} : \widehat{X} \rightarrow Y$ and this extension is $\widehat{\mathcal{U}}$ - \mathcal{V} uniformly continuous.*

4.2.2 General case

If (X, \mathcal{U}) is not Kolmogorov, we first compute its Kolmogorov quotient and then take the completion of this space.

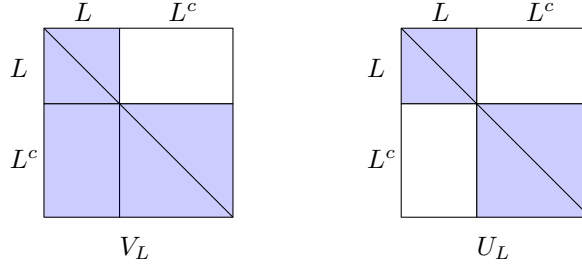
5 Pervin spaces

Pervin quasi-uniformities were first introduced in [8] to prove that every topological space can be derived from a quasi-uniform space. In this section, we give a thorough presentation of the properties of these spaces.

5.1 Definitions

For each subset L of a set X , consider the preorder V_L and the equivalence relation U_L on X :

$$\begin{aligned} V_L &= (X \times L) \cup (L^c \times X) = \{(x, y) \in X \times X \mid x \in L \implies y \in L\} \\ U_L &= (L \times L) \cup (L^c \times L^c) = \{(x, y) \in X \times X \mid x \in L \iff y \in L\} \end{aligned}$$



Let us prove immediately a few elementary, but useful, properties of these sets.

Lemma 5.1 *Let S and T be subsets of X . Then $S \times T \subseteq V_L$ if and only if $S \subseteq L^c$ or $T \subseteq L$ and $S \times T \subseteq U_L$ if and only if $S, T \subseteq L$ or $S, T \subseteq L^c$.*

Proof. Trivial (see the picture). \square

Proposition 5.2 *Let K and L be subsets of X .*

- (1) *One has $V_K \subseteq V_L$ if and only if $K = L$ and $U_K \subseteq U_L$ if and only if $K = L$ or $K = L^c$.*
- (2) *The set $V_K \cap V_L$ is a subset of both $V_{K \cap L}$ and $V_{K \cup L}$.*

Proof. (1) Observing that $V_L^c = L \times L^c$, one has $V_K \subseteq V_L$ if and only if $L \times L^c \subseteq K \times K^c$ and this latter condition holds if and only if $K = L$. The second part of the statement is clear.

(2) In the same way, we get

$$\begin{aligned} V_{K \cap L}^c &= (K \cap L) \times (K \cap L)^c = ((K \cap L) \times K^c) \cup ((K \cap L) \times L^c) \\ &\subseteq (K \times K^c) \cup (L \times L^c) = V_K^c \cup V_L^c = (V_K \cap V_L)^c \\ V_{K \cup L}^c &= (K \cup L) \times (K \cup L)^c = (K \times (K^c \cap L^c)) \cup (L \times (K^c \cap L^c)) \\ &\subseteq (K \times K^c) \cup (L \times L^c) = V_K^c \cup V_L^c = (V_K \cap V_L)^c \end{aligned}$$

and thus $V_K \cap V_L \subseteq V_{K \cap L}$ and $V_K \cap V_L \subseteq V_{K \cup L}$. \square

Let \mathcal{S} be a collection of subsets of X . The sets of the form $U_L [V_L]$, for $L \in \mathcal{S}$, form the subbasis of a [quasi]-uniformity of X , called the *Pervin [quasi]-uniformity defined by \mathcal{S}* and denoted by $\mathcal{U}_{\mathcal{S}} [V_{\mathcal{S}}]$. A [uniform] *Pervin space* is

a quasi-uniform [uniform] space of the form (X, \mathcal{V}_S) [(X, \mathcal{U}_S)] for some set \mathcal{S} of subsets of X . The next proposition shows that one can always assume that \mathcal{S} is a lattice (or a Boolean algebra in the uniform case).

Proposition 5.3 *Let \mathcal{S} be a collection of subsets of X and let \mathcal{L} be the lattice [Boolean algebra] of subsets generated by \mathcal{S} . Then the [uniform] Pervin spaces \mathcal{V}_S [\mathcal{U}_S] and $\mathcal{V}_\mathcal{L}$ [$\mathcal{U}_\mathcal{L}$] coincide.*

Proof. We prove the result for \mathcal{V}_S and $\mathcal{V}_\mathcal{L}$ only, the other case being similar. Since \mathcal{S} is contained in \mathcal{L} , every entourage of \mathcal{V}_S is an entourage of $\mathcal{V}_\mathcal{L}$. Let \mathcal{T} be the set of all subsets L of X such that V_L is an entourage of $\mathcal{V}_\mathcal{L}$. By definition, \mathcal{T} contains \mathcal{S} . We claim that \mathcal{T} is a lattice. Proposition 5.2 shows that $V_K \cap V_L \subseteq V_{K \cap L}$ and $V_K \cap V_L \subseteq V_{K \cup L}$. It follows that if V_K and V_L are entourages of \mathcal{V}_S , then so are $V_{K \cup L}$ and $V_{K \cap L}$, which proves the claim and shows that every entourage of $\mathcal{V}_\mathcal{L}$ is an entourage of \mathcal{V}_S . \square

The interest of working with a lattice or a Boolean algebra stems from the following results:

Proposition 5.4 *Let \mathcal{L} be a lattice [Boolean algebra] of subsets of X . Then the sets of the form V_L [U_L], for $L \in \mathcal{L}$, form a basis of $\mathcal{V}_\mathcal{L}$ [$\mathcal{U}_\mathcal{L}$].*

Proof. It follows immediately from Proposition 5.2. \square

Proposition 5.5 *Let \mathcal{L} be a lattice [Boolean algebra] of subsets of X and let L be a subset of X . Then L belongs to \mathcal{L} if and only if V_L [U_L] is an entourage of $\mathcal{V}_\mathcal{L}$ [$\mathcal{U}_\mathcal{L}$].*

Proof. We prove the result in the lattice case, the Boolean case being similar. If L belongs to \mathcal{L} , then V_L is by definition an entourage of $\mathcal{V}_\mathcal{L}$. Suppose now that V_L is an entourage of $\mathcal{V}_\mathcal{L}$. Then V_L contains a finite intersection $V_{L_1} \cap \dots \cap V_{L_n}$, where L_1, \dots, L_n are in \mathcal{L} . Taking complements yields the formula

$$L \times L^c \subseteq (L_1 \times L_1^c) \times \cup \dots \cup (L_n \times L_n^c) \quad (5.1)$$

We claim that

$$L = \bigcup_{I \in \mathcal{I}} \bigcap_{i \in I} L_i \quad (5.2)$$

where \mathcal{I} is the set of all subsets of $\{1, \dots, n\}$ for which there exists an element $v \in L$ such that $v \in L_i$ if and only if $i \in I$.

Let R be the right member of (5.2). Given $u \in L$, let $I = \{i \mid u \in L_i\}$. By construction, one gets $I \in \mathcal{I}$ and $u \in \bigcap_{i \in I} L_i$, whence $u \in R$. This proves the inclusion $L \subseteq R$. To prove the opposite direction, consider an element $u \in R$. By definition, there exists a set $I \in \mathcal{I}$ such that $u \in \bigcap_{i \in I} L_i$ and an element $v \in L$ such that $v \in L_i$ if and only if $i \in I$. Suppose that $u \notin L$. Then $(v, u) \in L \times L^c$ and (5.1) shows that $(v, u) \in L_i \times L_i^c$ for some i . Now, if $i \in I$, then $u \in L_i$, a contradiction. If $i \notin I$, then $v \notin L_i$ by definition of I , another contradiction. Therefore $u \in L$, which concludes the proof of (5.2) and shows that L belongs to \mathcal{L} . \square

An important consequence of Proposition 5.5 is the following characterisation of uniformly continuous functions between two Pervin spaces. Of course, a similar result holds for uniform Pervin spaces.

Theorem 5.6 *Let $X [Y]$ be a Pervin space defined by a lattice $\mathcal{K} [\mathcal{L}]$ of subsets of $X [Y]$. A function φ from X to Y is uniformly continuous if and only if, for every L in \mathcal{L} , $\varphi^{-1}(L)$ belongs to \mathcal{K} .*

Proof. The proof relies on the fact that if L is a subset of X and $K = \varphi^{-1}(L)$, then $(\varphi \times \varphi)^{-1}(V_L) = V_K$. Suppose that φ is uniformly continuous. If L is an element of \mathcal{L} , then V_L is an entourage of $\mathcal{V}_{\mathcal{L}}$ and thus V_K is an entourage of $\mathcal{V}_{\mathcal{K}}$. It follows by 5.5 that K belongs to \mathcal{K} .

Now assume that, for every L in \mathcal{L} , the set $\varphi^{-1}(L)$ belongs to \mathcal{K} . Then, for every L in \mathcal{L} , $(\varphi \times \varphi)^{-1}(V_L)$ is an entourage of $\mathcal{V}_{\mathcal{K}}$. Since the sets of the form V_L , for $L \in \mathcal{L}$, form a basis of $\mathcal{V}_{\mathcal{L}}$, the function φ is uniformly continuous. \square

5.2 The topology of a Pervin space

Let us look more closely at the topology of a Pervin space.

Proposition 5.7 *Let X be a Pervin space defined by a lattice [Boolean algebra] \mathcal{L} of subsets of X and let $x \in X$. Then*

- (1) *one has $\mathcal{N}(x) \cap \mathcal{L} = \{L \in \mathcal{L} \mid x \in L\}$.*
- (2) *the elements of \mathcal{L} form a basis of open [clopen] sets,*
- (3) *the elements of the form L^c , for $L \in \mathcal{L}$, form a basis of closed [clopen] sets.*

Proof. Observe that $V_L(x)$ is equal to L if x is in L and to X otherwise. It follows that an element of \mathcal{L} belongs to the filter of neighbourhoods of x if and only if it contains x . Consequently, the elements of \mathcal{L} form a basis of clopen [open] sets and the closed sets are the intersections of the sets of the form L^c , with $L \in \mathcal{L}$. \square

Let $(X, \mathcal{V}_{\mathcal{L}})$ be a Pervin space defined by a lattice \mathcal{L} of subsets of X . The uniform preorder can be simply defined in this case by $x \leq_{\mathcal{L}} y$ if and only if, for all $L \in \mathcal{L}$, $x \in L$ implies $y \in L$. It also admits another interesting description.

Proposition 5.8 *In a Pervin space, the uniform and the specialization preorders coincide.*

Proof. Let $\mathcal{V}_{\mathcal{L}}$ be a Pervin quasi-uniformity on X . The uniform preorder $\leq_{\mathcal{L}}$ is the intersection of all V_L , for $L \in \mathcal{L}$. It means that $x \leq_{\mathcal{L}} y$ if and only if, for all $L \in \mathcal{L}$, $x \in L$ implies $y \in L$. Since the elements of \mathcal{L} form a basis of open sets, this is equivalent to saying that y belongs to every open set that contains x . It follows that the uniform and the specialization preorders coincide. \square

We shall now forget about the qualifiers *uniform* and *specialization* and simply refer to *the* preorder of a Pervin space.

Corollary 5.9 *Any uniformly continuous map between two Pervin spaces is monotone.*

Example 5.1 (The space $\{0, 1\}$) Consider the uniform Pervin space on $\{0, 1\}$ generated by the entourages

$$V_{\{0\}} = \{(0, 0), (1, 1), (1, 0)\} \text{ and } V_{\{1\}} = \{(0, 0), (1, 1), (0, 1)\}$$

Since $V_{\{0\}} \cap V_{\{1\}}$ is equal to the diagonal, this uniformity is discrete and the associated topology is also discrete.

Example 5.2 (Sierpiński space) Consider the Pervin space on $\{0, 1\}$ generated by the entourage

$$V_{\{1\}} = \{(0, 0), (1, 1), (0, 1)\} = \{(x, y) \in \{0, 1\}^2 \mid x \leq y\}$$

The associated order is $0 \leq 1$. Its open sets are $\emptyset, \{1\}, \{0, 1\}$ and hence its closed sets are $\emptyset, \{0\}, \{0, 1\}$. The Sierpiński space is a Kolmogorov space but it is not accessible since $\{1\}$ is not closed.

5.3 A topological characterisation

The objective of this section is to show that the transitive and totally bounded spaces are exactly the Pervin spaces. Let us start with the easy part of this result.

Proposition 5.10 *Any Pervin space is transitive and totally bounded.*

Proof. A Pervin space $(X, \mathcal{U}_{\mathcal{L}})$ $[(X, \mathcal{U}_{\mathcal{L}})]$ is transitive since each relation $U_L [V_L]$ is transitive. It is also totally bounded, since U_L is contained in V_L and $L \cup L^c = X$. \square

The opposite direction is more difficult to prove and relies on the notion of block first introduced in [9]. A subset L of a [quasi]-uniform space X is a *block* if $U_L [V_L]$ is an entourage.

Blocks have a convenient description in terms of characteristic functions. Recall that the *characteristic function* of a subset L of a set X is the function χ_L from X to $\{0, 1\}$ defined by $\chi_L(x) = 1$ if $x \in L$ and $\chi_L(x) = 0$ if $x \in L^c$.

Proposition 5.11 *A subset of a [quasi]-uniform space is a block if and only if its characteristic function is a uniformly continuous function from X to the discrete uniform space $\{0, 1\}$ [to the Sierpiński space].*

Proof. We treat only the case of a quasi-uniform space, but the uniform case is similar. Let L be a subset of a quasi-uniform space X . Observe that $(x, y) \in V_L$ if and only if $\chi_L(x) \leq \chi_L(y)$, or, equivalently, if $(\chi_L(x), \chi_L(y)) \in V_{\{1\}}$. It follows that $V_L = (\chi_L \times \chi_L)^{-1}(V_{\{1\}})$. The result follows, since L is a block if and only if V_L is an entourage. \square

It is interesting to compare Proposition 5.11 with the following result:

Proposition 5.12 *Let X be a topological space and let L be a subset of X . Then L is open [clopen] if and only if its characteristic function is a continuous function from X to the Sierpiński [discrete] space $\{0, 1\}$.*

Proof. Indeed χ_L is continuous if and only if $\chi_L^{-1}(0)$ [and $\chi_L^{-1}(1)$] is [are] open. The result follows since $\chi_L^{-1}(0) = L$ and $\chi_L^{-1}(1) = L^c$. \square

The blocks of a Pervin space are easy to identify.

Proposition 5.13 *The blocks of the Pervin space defined by a Boolean algebra [lattice] \mathcal{L} are exactly the elements of \mathcal{L} .*

Proof. By Proposition 5.2 (2), the blocks of $\mathcal{U}_{\mathcal{L}}$ [$\mathcal{V}_{\mathcal{L}}$] form a Boolean algebra [lattice] containing \mathcal{L} . Conversely, if B is a block of $\mathcal{U}_{\mathcal{L}}$ [$\mathcal{V}_{\mathcal{L}}$], then U_B [V_B] is an entourage of $\mathcal{U}_{\mathcal{L}}$ [$\mathcal{V}_{\mathcal{L}}$]. Since the U_L [V_L], for $L \in \mathcal{L}$, form a basis of $\mathcal{U}_{\mathcal{L}}$ [$\mathcal{V}_{\mathcal{L}}$], each U_B [V_B] contains some U_L [V_L], with $L \in \mathcal{L}$. By Proposition 5.2 (1), this implies that $B = L$ or $B = L^c$ [$B = L$]. Thus B belongs to \mathcal{L} . \square

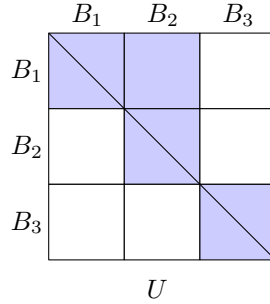
We can now establish the converse of Proposition 5.10. The argument of the first part of the proof is borrowed from [9].

Theorem 5.14 *A [quasi]-uniform space is a Pervin space if and only if it is transitive and totally bounded.*

Proof. We treat only the case of a quasi-uniform space, but the uniform case is similar. Let (X, \mathcal{U}) be a totally bounded, transitive quasi-uniform space. Let U be a transitive entourage of X . There exists a finite cover \mathcal{C} of X such that $\bigcup_{C \in \mathcal{C}} (C \times C) \subseteq U$. Let $B, C \in \mathcal{C}$. If $B \times C$ meets U , then $B \times C \subseteq U$ by transitivity. In particular, if both $B \times C$ and $C \times B$ have a nonempty intersection with U — for instance if $B \cap C \neq \emptyset$ —, then $(B \cup C) \times (B \cup C) \subseteq U$. So we may assume that \mathcal{C} is a partition of X , and that for all $B, C \in \mathcal{C}$, either $B \times C$ or $C \times B$ does not meet U . This uniquely determines the elements of the cover, which are called *U-blocks*. The *U*-blocks are ordered by the relation \leq defined by $B \leq C$ if $B \times C \subseteq U$ — or equivalently, if $B \times C$ meets U . Thus

$$U = \bigcup_{B \leq C} (B \times C). \quad (5.3)$$

The picture below shows the three blocks B_1, B_2, B_3 of an entourage U , with $B_1 \leq B_2$.



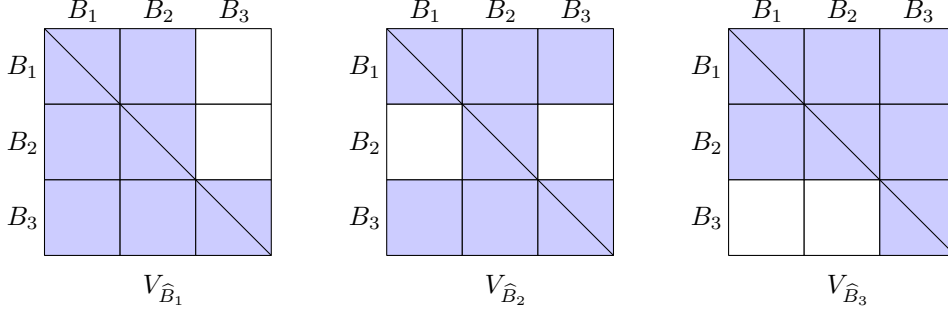
For each \leq -minimal *U*-block B , let \widehat{B} be the union of all blocks above B :

$$\widehat{B} = \bigcup_{B \leq C} C \quad (5.4)$$

In our example, one gets $\widehat{B}_1 = B_1 \cup B_2$, $\widehat{B}_2 = B_2$ and $\widehat{B}_3 = B_3$.

Lemma 5.15 *Each entourage U is the intersection of the sets $V_{\widehat{B}}$ such that B is a minimal *U*-block.*

For instance, one can see from the picture below that this property holds for our example since $U = V_{\widehat{B}_1} \cap V_{\widehat{B}_2} \cap V_{\widehat{B}_3}$.



Proof. Let us first show that each entourage $V_{\widehat{B}}$ contains U . Indeed, if $(c, d) \in U$, then by (5.3), one has $(c, d) \in C \times D$ for some U -blocks C and D such that $C \leq D$. If $B \leq C$, then $B \leq D$ by transitivity. It follows that either $B \not\leq C$ or $B \leq D$. Now by Lemma 5.1, $C \times D$ is a subset of $V_{\widehat{B}}$ if and only if $C \subseteq \widehat{B}^c$ or $D \subseteq \widehat{B}$, which, by the definition of \widehat{B} , means exactly that $B \not\leq C$ or $B \leq D$. Thus $C \times D$ is a subset of $V_{\widehat{B}}$, and U is a subset of $V_{\widehat{B}}$.

Suppose now that $(c, d) \notin V_{\widehat{B}}$ for some minimal U -block B . Let $C [D]$ be the U -block containing $c [d]$. Then $(C \times D) \cap V_{\widehat{B}} = \emptyset$, which means that $B \leq C$ and $B \not\leq D$. It follows that $C \not\leq D$, that is, $(C \times D) \cap U = \emptyset$. One gets finally $(c, d) \notin U$, which concludes the proof of the lemma. \square

We can now conclude the proof of the theorem. Let \mathcal{L} be the lattice generated by the sets of the form \widehat{B} , where B is a minimal U -block of some entourage U of X . By Lemma 5.15, each set \widehat{B} contains U and thus is an entourage of \mathcal{U} . The same lemma shows that U is an intersection of entourages of $\mathcal{V}_{\mathcal{L}}$. Therefore $\mathcal{V}_{\mathcal{L}} = \mathcal{U}$, and thus X is a Pervin space. \square

The quotient of a Pervin space is not necessarily a Pervin space. Consider for instance the metric $d(x, y) = |x - y|$ on the space $[0, 1]$. Let \mathcal{U} be the uniformity defined by d and let $\mathcal{U}_{\mathcal{L}}$ be the Pervin uniformity defined by the lattice of open sets of $[0, 1]$. We claim that \mathcal{U} is contained in $\mathcal{U}_{\mathcal{L}}$. Indeed, let $\varepsilon > 0$ and let n be an integer such that $n\varepsilon \geq 1$. Then if $(x, y) \in U_{[0, \varepsilon[} \cap U_{[\varepsilon, 2\varepsilon[} \cap \dots \cap U_{[(n-1)\varepsilon, 1]}$, then $d(x, y) < \varepsilon$. It follows that the identity from $([0, 1], \mathcal{U}_{\mathcal{L}})$ into $([0, 1], \mathcal{U})$ is uniformly continuous. However $([0, 1], \mathcal{U})$ is not a Pervin space.

However, a weaker property holds.

Corollary 5.16 *A transitive quotient of a Pervin space is a Pervin space.*

Proof. By Proposition 4.2, the quotient of a totally bounded quasi-uniform space is totally bounded. The result is now an immediate consequence of Theorem 5.14. \square

La fin de cette section doit être mise ailleurs.

Intuitively, blocks are to uniformities what clopen sets are to topologies. Pervin uniformities are the uniform analogs of totally discontinuous topologies: a topology is totally discontinuous [a uniformity is Pervin] if and only if it has a basis of clopen sets [blocks].

Proposition 5.17 *The blocks of a [quasi]-uniform space form a Boolean sub-algebra [sublattice] of the Boolean algebra [lattice] formed by the clopen [open] sets. These two Boolean algebras [lattices] coincide if the space is compact.*

Proof. Proposition 5.2 insures that the blocks form a Boolean algebra [lattice]. Proposition 5.11 shows that the blocks are clopen [open], since $\{1\}$ is a clopen [open] set of the discrete uniform space $\{0, 1\}$ [Sierpiński space]. In the compact case, use the fact that continuity and uniform continuity are equivalent. \square

A topological space is *zero-dimensional* if it has a basis consisting of clopen subsets. It is *totally disconnected* if its connected components are singletons. It is well known that a compact space is zero-dimensional if and only if it is totally disconnected.

5.4 Completion of a Pervin space

The completion of a Pervin space was first studied by Császár [3]. However, Császár considered another type of completion, the D -completion, which fortunately leads to the same completion as ours in the case of a Pervin space. But we are really indebted to this little known paper and a number of results of this section are merely a reformulation of Császár's results.

Let us consider a Pervin space $(X, \mathcal{V}_{\mathcal{L}})$ defined by a lattice \mathcal{L} of subsets of X . We also denote by \mathcal{B} the Boolean algebra generated by \mathcal{L} . Cauchy filters have a simple characterisation.

Proposition 5.18 *A filter \mathcal{F} on X is Cauchy if and only if, for every $L \in \mathcal{L}$, either $L \in \mathcal{F}$ or $L^c \in \mathcal{F}$.*

Proof. Let \mathcal{F} be a Cauchy filter and let $L \in \mathcal{L}$. Since V_L is an entourage, there exists an F in \mathcal{F} with $F \times F \subseteq V_L$. Since $\mathcal{V}_L = (X \times L) \cup (L^c \times X)$ one gets either $F \subseteq L$ or $F \subseteq L^c$. Therefore, either $L \in \mathcal{F}$ or $L^c \in \mathcal{F}$.

Suppose that, for every $L \in \mathcal{L}$, either $L \in \mathcal{F}$ or $L^c \in \mathcal{F}$. Let V be an entourage of $\mathcal{V}_{\mathcal{L}}$. By Proposition 5.4, V contains an entourage of the form V_L . Since one has $L \times L \subseteq V_L$ and $L^c \times L^c \subseteq V_L$, it follows that \mathcal{F} is Cauchy. \square

In a Pervin space, one has $V_L^s = U_L$. It follows by Proposition 5.3 that $\mathcal{V}_{\mathcal{L}}^s = \mathcal{U}_{\mathcal{B}} = \mathcal{U}_{\mathcal{B}}$. Recall that the completion \widehat{X} of X is the set of minimal $\mathcal{U}_{\mathcal{B}}$ -Cauchy filters on X .

Our next objective is to show that \widehat{X} can be identified with the prime spectrum of \mathcal{L} . This can be done by extending Nerode's trick as follows. If \mathcal{F} is a filter of \mathcal{L} , let $\mathcal{F}^{\#}$ be the filter of X generated by $\mathcal{F} \cup \{L^c \mid L \in \mathcal{L} - \mathcal{F}\}$. Thus

$$\mathcal{F}^{\#} = \{S \subseteq X \mid \text{there exist } F \in \mathcal{F} \text{ and } L \in \mathcal{L} - \mathcal{F} \text{ such that } F \cap L^c \subseteq S\}.$$

Note that if \mathcal{L} is a Boolean algebra, the definition simplifies to

$$\mathcal{F}^{\#} = \{S \subseteq X \mid \text{there exists } F \in \mathcal{F} \text{ such that } F \subseteq S\}$$

but we also need the definition in the lattice case.

Proposition 5.19 *If \mathcal{F} is a prime filter of \mathcal{L} , then $\mathcal{F}^{\#}$ is a minimal Cauchy filter of (X, \mathcal{U}) and $\mathcal{F}^{\#} \cap \mathcal{L} = \mathcal{F}$.*

Proof. Let \mathcal{F} is a prime filter of \mathcal{L} . We first prove that \mathcal{F}^\sharp is a Cauchy filter. By construction, \mathcal{F}^\sharp satisfies (F₁) and (F₂). If $S_1, S_2 \in \mathcal{F}^\sharp$, there exist $G_1, G_2 \in \mathcal{F}$ and $L_1, L_2 \in \mathcal{L} - \mathcal{F}$ such that $G_1 \cap L_1^c \subseteq S_1$ and $G_2 \cap L_2^c \subseteq S_2$. Thus $G_1 \cap G_2 \in \mathcal{F}$ and $L_1 \cup L_2 \in \mathcal{L} - \mathcal{F}$ since \mathcal{F} is prime and the formula

$$(G_1 \cap G_2) \cap (L_1 \cup L_2)^c = ((G_1 \cap G_2) \cap L_1^c) \cup ((G_1 \cap G_2) \cap L_2^c) \subseteq S_1 \cap S_2$$

shows that $S_1 \cap S_2$ belongs to \mathcal{F}^\sharp . Thus \mathcal{F}^\sharp is a filter. Further, if $L \in \mathcal{L}$, then either $L \in \mathcal{F}$ or $L \in \mathcal{L} - \mathcal{F}$. In the first case, $L \in \mathcal{F}^\sharp$ and in the latter case, $L^c \in \mathcal{F}^\sharp$. It follows by Proposition 5.18 that \mathcal{F}^\sharp is a Cauchy filter.

Let $L \in \mathcal{L}$. If $L \in \mathcal{F}$, then $L \in \mathcal{F}^\sharp$ since $L \cap \emptyset^c \subseteq L$ and $\emptyset \in \mathcal{L} - \mathcal{F}$. If $L \notin \mathcal{F}$, then $L^c \in \mathcal{F}^\sharp$ since $L^c = X \cap L^c$, $X \in \mathcal{F}$ and $L \in \mathcal{L} - \mathcal{F}$. Thus $L \notin \mathcal{F}^\sharp$. Therefore, one has $\mathcal{F}^\sharp \cap \mathcal{L} = \mathcal{F}$.

Finally, suppose that \mathcal{G} is a Cauchy filter contained in \mathcal{F}^\sharp . Let $L \in \mathcal{L}$. If $L \in \mathcal{G}$, then $L \in \mathcal{F}^\sharp$. If $L \notin \mathcal{G}$, then by Proposition 5.18, $L^c \in \mathcal{G}$ and thus $L^c \in \mathcal{F}^\sharp$ and $L \notin \mathcal{F}^\sharp$. It follows that $\mathcal{G} \cap \mathcal{L} = \mathcal{F}^\sharp \cap \mathcal{L} = \mathcal{F}$. Let $S \in \mathcal{F}^\sharp$. Then there exist $F \in \mathcal{F}$ and $L \in \mathcal{L} - \mathcal{F}$ such that $F \cap L^c \subseteq S$. Thus $F \in \mathcal{G}$, $L \notin \mathcal{G}$ and $L^c \in \mathcal{G}$ by Proposition 5.18. It follows that $S \in \mathcal{G}$ and thus $\mathcal{G} = \mathcal{F}^\sharp$. Therefore, \mathcal{F}^\sharp is a minimal Cauchy filter. \square

The connection between the minimal Cauchy filters on X and the prime filters of \mathcal{L} is very similar to Proposition 1.2.

Theorem 5.20 *The maps $\mathcal{F} \mapsto \mathcal{F} \cap \mathcal{L}$ and $\mathcal{F} \mapsto \mathcal{F}^\sharp$ define mutually inverse bijections between the set of minimal Cauchy filters on $(X, \mathcal{U}_{\mathcal{B}})$ and the set of prime filters of \mathcal{L} .*

Proof. Proposition 5.19 already shows that if \mathcal{F} is a prime filter of \mathcal{L} , then \mathcal{F}^\sharp is a minimal Cauchy filter and $\mathcal{F}^\sharp \cap \mathcal{L} = \mathcal{F}$.

Let \mathcal{F} be a minimal Cauchy filter on X and let $K, L \in \mathcal{L}$. Suppose that $K \cup L \in \mathcal{F}$. We claim that either K or L is in \mathcal{F} . Otherwise, by Proposition 5.18, \mathcal{F} contains both K^c and L^c and hence also $K^c \cap L^c$. This is a contradiction, since $K^c \cap L^c = (K \cup L)^c$. This proves the claim and shows that $\mathcal{F} \cap \mathcal{L}$ is a prime filter. Let us show that $(\mathcal{F} \cap \mathcal{L})^\sharp$ is contained in \mathcal{F} . Let $S \in (\mathcal{F} \cap \mathcal{L})^\sharp$. Then there exist $F \in \mathcal{F} \cap \mathcal{L}$ and $L \in \mathcal{L} - \mathcal{F}$ such that $F \cap L^c \subseteq S$. Since \mathcal{F} is a Cauchy filter, $L^c \in \mathcal{F}$ by Proposition 5.18 and thus $S \in \mathcal{F}$ as required. Since \mathcal{F} is a minimal Cauchy filter and $(\mathcal{F} \cap \mathcal{L})^\sharp$ is a Cauchy filter, these two filters are equal, that is $(\mathcal{F} \cap \mathcal{L})^\sharp = \mathcal{F}$. This concludes the proof. \square

Theorem 5.20 shows that one can identify \widehat{X} with the set of prime filters of \mathcal{L} . If we do so, then the embedding ι of X into \widehat{X} is given by

$$\iota(x) = \mathcal{N}(x) = \{L \in \mathcal{L} \mid x \in L\}$$

Let us now describe the quasi-uniformity on \widehat{X} .

Proposition 5.21 *The sets*

$$\widehat{V}_L = \{(\mathcal{F}, \mathcal{G}) \mid L \in \mathcal{F} \text{ implies } L \in \mathcal{G}\}$$

for $L \in \mathcal{L}$, form a basis for the quasi-uniformity on \widehat{X} .

Proof. By definition, the sets

$$\widehat{V} = \{(\mathcal{F}, \mathcal{G}) \mid \text{there exists } F \in \mathcal{F} \text{ and } G \in \mathcal{G} \text{ such that } F \times G \subseteq V\}$$

where V is an entourage of $\mathcal{V}_{\mathcal{L}}$, form a basis for the quasi-uniformity on \widehat{X} . Since the sets V_L , for $L \in \mathcal{L}$, form a basis of $\mathcal{V}_{\mathcal{L}}$, the sets \widehat{V}_L form a basis for the quasi-uniformity on \widehat{X} . Now

$$\widehat{V}_L = \{(\mathcal{F}, \mathcal{G}) \mid L^c \in \mathcal{F} \text{ or } L \in \mathcal{G}\} = \{(\mathcal{F}, \mathcal{G}) \mid L \in \mathcal{F} \text{ implies } L \in \mathcal{G}\}$$

The second part of the proposition follows immediately. \square

We now analyse the consequences of this result on the topology on \widehat{X} . For each $L \in \mathcal{L}$, let us set

$$\begin{aligned} O_L &= \{\mathcal{F} \in \widehat{X} \mid L \in \mathcal{F}\} \\ C_L &= \{\mathcal{F} \in \widehat{X} \mid L \notin \mathcal{F}\} \end{aligned}$$

Corollary 5.22 *The sets of the form O_L , for $L \in \mathcal{L}$, form a basis of open sets of \widehat{X} . The sets of the form C_L , for $L \in \mathcal{L}$, form a basis of closed sets of \widehat{X} .*

Proof. By Proposition 5.18, it suffices to prove the first part of the statement. By definition, the sets of the form $\widehat{V}_L(\mathcal{F})$, for $L \in \mathcal{L}$ and $\mathcal{F} \in \widehat{X}$, form a basis of the topology of \widehat{X} . Since

$$\widehat{V}_L(\mathcal{F}) = \begin{cases} \{\mathcal{G} \in \widehat{X} \mid L \in \mathcal{G}\} & \text{if } L \in \mathcal{F}, \\ \widehat{X} & \text{otherwise,} \end{cases}$$

the result follows. \square

The sets O_L have some interesting properties. Let us start with an easy one.

Proposition 5.23 *Let $(L_i)_{i \in F}$ be a finite family of elements of \mathcal{L} and let $L = \bigcup_{i \in F} L_i$. Then $O_L = \bigcup_{i \in F} O_{L_i}$.*

Proof. Let \mathcal{F} be a prime filter in O_L . Then $L \in \mathcal{F}$ and since $L = \bigcup_{i \in F} L_i$, one of the L_i belongs to \mathcal{F} . Thus \mathcal{F} belongs to $\bigcup_{i \in F} O_{L_i}$. Suppose now that \mathcal{F} belongs to $\bigcup_{i \in F} O_{L_i}$. Then \mathcal{F} belongs to some O_{L_i} , that is, some L_i belongs to \mathcal{F} . Since $L_i \subseteq L$, it follows that $L \in \mathcal{F}$ and hence \mathcal{F} belongs to O_L . Thus $O_L = \bigcup_{i \in F} O_{L_i}$. \square

The next property is less trivial.

Theorem 5.24 *A subset of \widehat{X} is compact open if and only if it is of the form O_L for some $L \in \mathcal{L}$.*

Proof. Let K be a compact open subset of \widehat{X} . Then for each $x \in K$, there is an open set O_{L_x} containing x and contained in K . It follows that the family $(O_{L_x})_{x \in K}$ covers K and admits a finite subcover $(O_{L_i})_{i \in F}$. Setting $L = \bigcup_{i \in F} L_i$, we get by Proposition 5.23 $K = \bigcup_{i \in F} O_{L_i} = O_L$.

Let $L \in \mathcal{L}$. We already know that O_L is open. To prove that it is compact, consider a covering of O_L by a family of open sets. By Corollary 5.22, each open

set is itself a union of sets of the form O_M , with $M \in \mathcal{L}$. This gives a covering of O_L of the form $(O_{L_i})_{i \in I}$, where each L_i belongs to \mathcal{L} . Since O_L is the set of prime filters of \mathcal{L} containing L , this simply means that every prime filter containing L contains one L_i , for some $i \in I$. Let \mathcal{I} be the ideal of \mathcal{L} generated by the L_i , for $i \in I$. If L does not belong to \mathcal{I} , then by Theorem 1.3, there is a prime filter containing L but containing no L_i , a contradiction. Thus L belongs to \mathcal{I} , that is, L is a subset of some finite union $\bigcup_{i \in F} L_i$. Consequently, every prime filter containing L contains one L_i for some $i \in F$. In other words, $(O_{L_i})_{i \in F}$ is a finite cover of O_L and thus O_L is compact. \square

Corollary 5.25 *The space \widehat{X} is compact.*

Proof. It suffices to apply Theorem 5.24 with $L = X$. \square

Let us now assume that the preorder on X is an order (or equivalently, that X is Kolmogorov). Recall that, in this case, the map $x \mapsto \mathcal{N}(x)$ is a uniformly continuous embedding from X into \widehat{X} . We tacitly make use of this embedding to identify X with a subset of \widehat{X} .

Proposition 5.26 *For each subset S of X , one has*

$$\overline{S} = \bigcap_{\{K \in \mathcal{L} \mid K \subseteq S^c\}} C_K \quad (5.5)$$

and

$$\overline{S} \cap X = \bigcap_{\{K \in \mathcal{L} \mid S \subseteq K^c\}} K^c \quad (5.6)$$

Proof. By definition, \overline{S} is the intersection of all sets C_K such that, for each $x \in S$, the filter $\mathcal{N}(x)$ is in C_K . This condition means that for all $x \in S$, x is in K^c . It is equivalent to say that S is a subset of K^c , or that K is a subset of S^c , which leads to Formula (5.5).

Let $x \in X$. By Formula (5.5), one has $\mathcal{N}(x) \in \overline{S}$ if and only if, for each $K \in \mathcal{L}$ with $K \subseteq S^c$, one has $K \notin \mathcal{N}(x)$, that is, $x \in K^c$. Formula 5.6 follows immediately. \square

Let us make precise that in the next statement, $L^c = X - L$ and that $\overline{L^c} = \widehat{X} - \overline{L}$. Therefore, $\overline{L^c}$ is not the interior of L as one may expect from the notation.

Corollary 5.27 *For each $L \in \mathcal{L}$, the following formulas hold:*

$$\begin{aligned} \overline{L^c} &= C_L = \widehat{\chi}_L^{-1}(0) & \overline{L^c}^c &= O_L = \widehat{\chi}_L^{-1}(1) \\ \overline{L^c} \cap X &= C_L \cap X = L^c & \overline{L^c}^c \cap X &= O_L \cap X = L. \end{aligned}$$

Proof. Formula (5.5) applied to $S = L^c$ gives

$$\overline{L^c} = \bigcap_{\{K \in \mathcal{L} \mid L^c \subseteq K^c\}} C_K = \bigcap_{\{K \in \mathcal{L} \mid K \subseteq L\}} C_K = C_L.$$

Taking complements, we get $\overline{L^c}^c = O_L$. Formula 5.6 applied to $S = L^c$ gives

$$\overline{L^c} \cap X = \bigcap_{\{K \in \mathcal{L} \mid L^c \subseteq K^c\}} K^c = L^c$$

It follows that $\overline{L^c} \cap X = L$. Finally, since $\chi_L(L^c) = 0$ by definition, one has $\widehat{\chi}_L(\overline{L^c}) = 0$. Further, since X is dense in \widehat{X} , $O_L \cap X$ is dense in O_L and since $\overline{L^c} = O_L$ one gets

$$\widehat{\chi}_L(\overline{L^c}) = \widehat{\chi}_L(O_L) = \widehat{\chi}_L(O_L \cap X) = \widehat{\chi}_L(L) = \chi_L(L) = 1$$

It follows that $\overline{L^c} = \widehat{\chi}_L^{-1}(1)$ and thus $\overline{L^c} = \widehat{\chi}_L^{-1}(0)$. \square

Theorem 5.28 *The maps $L \mapsto \overline{L^c}$ and $K \mapsto K \cap X$ are mutually inverse lattice isomorphisms between \mathcal{L} and the set of compact-open subsets of \widehat{X} .*

Proof. Corollary 5.27 shows that if $L \in \mathcal{L}$, then $\overline{L^c}$ is compact open and equal to O_L . By Theorem 5.24, each compact open set is of the form O_L for some L and Corollary 5.27 shows that $O_L \cap X = L$. Consequently, the maps $L \mapsto \overline{L^c}$ and $K \mapsto K \cap X$ are mutually inverse bijections between \mathcal{L} and the set of compact-open subsets of \widehat{X} .

Proposition 5.23 shows that $O_{L_1} \cup O_{L_2} = O_{L_1 \cup L_2}$. Further, Corollary 5.27 gives

$$O_{L_1} \cap O_{L_2} = \overline{L_1^c} \cap \overline{L_2^c} = (\overline{L_1^c} \cup \overline{L_2^c})^c = (\overline{L_1^c \cup L_2^c})^c = \overline{(L_1 \cap L_2)^c}^c = O_{L_1 \cap L_2}.$$

It follows that the map $L \mapsto \overline{L^c}$ is a lattice isomorphism. \square

Thus the sets of the form O_L are exactly the compact open sets of \widehat{X} and these sets form a lattice isomorphic to \mathcal{L} .

Corollary 5.29 *The completion of a Pervin space $(X, \mathcal{V}_{\mathcal{L}})$ is the Pervin space on \widehat{X} defined by the lattice of its compact open sets.*

Proof. Proposition 5.21 shows that the sets \widehat{V}_L , for $L \in \mathcal{L}$, form a basis for the quasi-uniformity on \widehat{X} . It suffices now to observe that since $\mathcal{F} \in O_L$ if and only if $L \in \mathcal{F}$, one has $\widehat{V}_L = \{(\mathcal{F}, \mathcal{G}) \mid \mathcal{F} \in O_L \text{ implies } \mathcal{G} \in O_L\}$. \square

In particular, the uniform and the specialization preorders coincide on \widehat{X} .

5.5 The Boolean case

If \mathcal{L} is a Boolean algebra, the results of the previous sections simplify greatly. First, the uniform (or specialization) preorders on X and on \widehat{X} are equivalence relations. Next, $O_L = C_{L^c}$ and thus the O_L form a basis of clopen sets. Further, for each $L \in \mathcal{L}$, one has $\overline{L} = O_L = \widehat{\chi}_L^{-1}(1)$ and $\overline{L} \cap X = L$. Finally, Theorem 5.28 can be restated as follows.

Theorem 5.30 *The maps $L \mapsto \overline{L}$ and $K \mapsto K \cap X$ are mutually inverse Boolean algebra isomorphisms between \mathcal{L} and the set of clopen subsets of \widehat{X} .*

5.6 Summary

We gave three equivalent descriptions of the completion of the Pervin space $(X, \mathcal{V}_{\mathcal{L}})$: \widehat{X} can be identified to the set of minimal Cauchy filters on X , or to the prime spectrum of \mathcal{L} , or yet to the set of valuations on \mathcal{L} . The table below can serve as a dictionary to pass from one description to the other:

\widehat{X}	Minimal Cauchy filters on X	Prime filters of \mathcal{L}	Valuations on \mathcal{L}
Order	$\mathcal{F} \leq \mathcal{F}'$ iff for $L \in \mathcal{L}$ $L \in \mathcal{F} \implies L \in \mathcal{F}'$	Inclusion	$v \leq v'$ iff for $L \in \mathcal{L}$ $v(L) \leq v'(L)$
Entourages \widehat{V}_L	$\{(\mathcal{F}, \mathcal{F}') \mid L \in \mathcal{F} \implies L \in \mathcal{F}'\}$	$\{(\mathcal{F}, \mathcal{F}') \mid L \in \mathcal{F} \implies L \in \mathcal{F}'\}$	$\{(v, v') \mid v(L) = 1 \implies v'(L) = 1\}$
$\iota(x)$	$\{L \in \mathcal{L} \mid x \in L\}^\#$	$\{L \in \mathcal{L} \mid x \in L\}$	v such that $v(L) = 1$ iff $x \in L$
O_L	{ Cauchy filters containing L }	{ prime filters containing L }	$\{v \mid v(L) = 1\}$
C_L	{ Cauchy filters not containing L }	{ prime filters not containing L }	$\{v \mid v(L) = 0\}$

Another construction of the completion.

[James, p. 149, p. 155] Let X be a metric space. Then the space C of all continuous functions from X into $[0, 1]$ is a complete metric space for the supremum metric. One can embed X into $C(X)$ by fixing a point x_0 of X and by considering the function $\varphi : X \rightarrow C$ defined by $\varphi(x)(u) = d(u, x) - d(u, x_0)$. Then X is dense in the closure of $\varphi(X)$ and this closure is the completion of X . This method of constructing the metric completion does not appear to extend to uniform spaces in general.

Stone-Čech compactification. The map from X into $[0, 1]^C$ which maps x onto $(f(x))_{f \in C}$ is a continuous map onto its image, and the closure of X is the Stone-Čech compactification of X .

Let \mathcal{L} be a lattice of subsets of X and let $\{0, 1\}$ be the Sierpinski space. Since it is complete, the space $\{0, 1\}^{\mathcal{L}}$ is also a complete quasi-uniform space. Further, the set of valuations on \mathcal{L} is a closed subset of $\{0, 1\}^{\mathcal{L}}$ and hence, it is a complete quasi-uniform space.

6 Duality results

Duality theory yields three closely related representations of bounded distributive lattices via Priestley spaces, spectral spaces, and pairwise Stone spaces.

In this section, we show that Pervin spaces provides a fourth representation for lattices of subsets of a given set. Taking the completion of these Pervin spaces gives back the three other dualities.

6.1 Duality between lattices and Pervin spaces

Let \mathcal{L} be a lattice [Boolean algebra] of subsets of a set X . The objective of this section is to establish a duality between the sublattices of \mathcal{L} and the transitive quotients of the Pervin space $(X, \mathcal{V}_{\mathcal{L}})$. Let us state this result in a more precise way.

Theorem 6.1 *If \mathcal{K} is a sublattice of \mathcal{L} , then $(X, \mathcal{V}_{\mathcal{K}})$ is a transitive quotient of $(X, \mathcal{V}_{\mathcal{L}})$. If Y is a transitive quotient of $(X, \mathcal{V}_{\mathcal{L}})$, then there is a sublattice \mathcal{K} of \mathcal{L} such that $(X, \mathcal{V}_{\mathcal{K}})$ is isomorphic to Y .*

Proof. If \mathcal{K} is a sublattice of \mathcal{L} , it follows from Theorem 5.6 that the identity on X is $(\mathcal{V}_{\mathcal{L}}, \mathcal{V}_{\mathcal{K}})$ -uniformly continuous. Thus $(X, \mathcal{V}_{\mathcal{K}})$ is a transitive quotient of $(X, \mathcal{V}_{\mathcal{L}})$.

Let now Y be a transitive quotient of $(X, \mathcal{V}_{\mathcal{L}})$. Proposition 4.2 shows that Y is totally bounded. It follows from Theorem 5.14 that Y is a Pervin space. Let \mathcal{M} be a lattice of subsets of Y defining the quasi-uniformity on Y and let $\varphi : X \rightarrow Y$ be the surjective uniformly continuous map defining the quotient. Theorem 5.6 shows that if $M \in \mathcal{M}$, then $\varphi^{-1}(M)$ belongs to \mathcal{L} . It follows that φ^{-1} induces a lattice isomorphism between \mathcal{M} and $\varphi^{-1}(\mathcal{M})$ and hence \mathcal{M} is isomorphic to a sublattice of \mathcal{L} . \square

6.2 Stone and Priestley dualities

We shall now recover a well-known result of duality theory, which says, in essence, that there is a bijection between the sublattices of \mathcal{L} and the dual space of \mathcal{L} . In this statement, \mathcal{L} denotes the completion of the Pervin space $(X, \mathcal{V}_{\mathcal{L}})$.

Proposition 6.2 *Let \mathcal{K} is a sublattice of \mathcal{L} . The identity on X is $(\mathcal{V}_{\mathcal{L}}, \mathcal{V}_{\mathcal{K}})$ -uniformly continuous and extends to a surjective uniformly continuous map from $\widehat{X}^{\mathcal{L}}$ to $\widehat{X}^{\mathcal{K}}$.*

Proof. For this proof, it is convenient to view the completion of $(X, \mathcal{V}_{\mathcal{L}})$ as the set of valuations on \mathcal{L} . It follows immediately from Theorem 5.6 that the identity on X is $(\mathcal{V}_{\mathcal{L}}, \mathcal{V}_{\mathcal{K}})$ -uniformly continuous. Therefore, it has a unique uniformly continuous extension ρ from $\widehat{X}^{\mathcal{L}}$ to $\widehat{X}^{\mathcal{K}}$. We claim that ρ maps a valuation on \mathcal{L} to its restriction to \mathcal{K} . Indeed, an element x of X is identified in $\widehat{X}^{\mathcal{L}}$ [$\widehat{X}^{\mathcal{K}}$] with the valuation v on \mathcal{L} [\mathcal{K}] defined by $v(L) = 1$ if and only if $x \in L$ [$x \in K$]. It follows that the restriction of ρ to X is the identity. Further, ρ is uniformly continuous since, if $K \in \mathcal{K}$, then $(\rho \times \rho)^{-1}(\widehat{V}_K) = \widehat{V}_K$ and \widehat{V}_K is an entourage of $\widehat{X}^{\mathcal{L}}$, since $\mathcal{K} \subseteq \mathcal{L}$. This proves the claim. Further ρ is surjective by construction and thus $\widehat{X}^{\mathcal{K}}$ is a quotient of $\widehat{X}^{\mathcal{L}}$. \square

Proposition 6.3 *If a Kolmogorov quasi-uniform space Y is a quotient of $(X, \mathcal{V}_{\mathcal{L}})$, then there is a sublattice \mathcal{K} of \mathcal{L} such that Y is isomorphic to $(X, \mathcal{V}_{\mathcal{K}})$.*

Proof.

Theorem 6.4 *If $(X, \mathcal{V}_{\mathcal{L}})$ is Kolmogorov, then there is a duality between the sublattices of \mathcal{L} and the Kolmogorov quotients of $(X, \mathcal{V}_{\mathcal{L}})$.*

Proposition 6.5 *If a complete Kolmogorov quasi-uniform space Y is a quotient of $\tilde{X}^{\mathcal{L}}$, then there is a sublattice \mathcal{K} of \mathcal{L} such that Y is the completion of $(X, \mathcal{V}_{\mathcal{K}})$.*

Proof. TO DO. \square

Corollary 6.6 *Given a distributive lattice \mathcal{L} , there is a maximal Kolmogorov Pervin space defined by \mathcal{L} .*

6.3 P -metrics

There are numerous variations possible to the notion of a metric. The notion we consider in this section is well suited for the study of Pervin spaces. We call it a P -metric, where P refers to Pervin, to avoid the accumulation of prefixes like pseudo, quasi, semi or ultra used in the literature¹.

A P -metric d on a set X is a mapping $d: X \times X \rightarrow \mathbb{R}^+$ satisfying the following conditions:

(P₁) for all $x \in X$, $d(x, x) = 0$.

(P₂) for all $x, y, z \in X$, $d(x, z) \leq \max(d(x, y), d(y, z))$.

A P -metric d on a set X naturally defines a transitive quasiuniformity \mathcal{U} on X . A basis of \mathcal{U} is given by the entourages of the form

$$B_n = \{(x, y) \in X \times X \mid d(x, y) < 2^{-n}\}, \text{ where } n \text{ is a nonnegative integer.}$$

The Pervin spaces defined by a P -metric have a simple characterisation.

Proposition 6.7 *Let \mathcal{L} be a lattice of subsets of a set X . The following conditions are equivalent:*

- (1) *the quasiuniformity $\mathcal{V}_{\mathcal{L}}$ can be defined by a P -metric,*
- (2) *the quasiuniformity $\mathcal{V}_{\mathcal{L}}$ has a countable basis,*
- (3) *the lattice \mathcal{L} is countable.*

If these conditions are satisfied, the quasiuniformity of the completion of $(X, \mathcal{V}_{\mathcal{L}})$ can also be defined by a P -metric.

Proof. (1) implies (2) since the $(B_n)_{n \in \mathbb{N}}$ form a countable basis.

(2) implies (1). Let $(V_n)_{n \in \mathbb{N}}$ be a countable basis of $\mathcal{V}_{\mathcal{L}}$. We define a P -metric d on X as follows. For each $n \geq 0$, let $W_n = \bigcap_{0 \leq k \leq n} V_k$ and

$$d(x, y) = \begin{cases} 2^{-n} & \text{if } n \text{ is the smallest integer such that } (x, y) \notin W_n, \\ 0 & \text{if } (x, y) \in W_n \text{ for all } n \geq 0. \end{cases}$$

Then $d(x, y) < 2^{-n}$ if and only if $(x, y) \in W_n$. Therefore, d defines exactly the quasiuniformity $\mathcal{V}_{\mathcal{L}}$.

(3) implies (2) since, by Proposition 5.4, the sets V_L form a basis of $\mathcal{V}_{\mathcal{L}}$.

(2) implies (3). Let $(V_n)_{n \in \mathbb{N}}$ be a countable basis for $\mathcal{V}_{\mathcal{L}}$ and let $L \in \mathcal{L}$. Then $V_{n_L} \subseteq V_L$ for some $n_L \in \mathbb{N}$, and since the family $(V_K)_{K \in \mathcal{L}}$ is a basis of $\mathcal{V}_{\mathcal{L}}$, one also has $V_K \subseteq V_{n_L}$ for some $K \in \mathcal{L}$. Therefore $V_K \subseteq V_{n_L} \subseteq V_L$ and

¹According to the literature, it should be called a *nonarchimedean pseudo quasi-metric*, a really complicated name for a notion defined by two simple conditions.

finally $K = L$ by Proposition 5.2. It follows that $V_L = V_{n_L}$ and hence the map $L \mapsto n_L$ is injective. Consequently, \mathcal{L} is countable.

Corollary 5.29 and Theorem 5.24 show that the completion \widehat{X} of $(X, \mathcal{V}_{\mathcal{L}})$ is a Pervin space defined by the lattice $(O_L)_{L \in \mathcal{L}}$. If \mathcal{L} is countable, this lattice is also countable. It follows by the first part of the proof that the quasiuniformity of \widehat{X} can be defined by a P -metric. \square

If \mathcal{L} is countable, then the topology induced by $\mathcal{V}_{\mathcal{L}}$ also has a countable basis. Indeed, by Proposition 5.7, the elements of \mathcal{L} form a basis of open sets. But this latter condition does not imply that \mathcal{L} is countable. For instance, let $X = \mathbb{N}$ and $\mathcal{L} = \mathcal{P}(\mathbb{N})$. Then \mathcal{L} is not countable but the topology of $(X, \mathcal{V}_{\mathcal{L}})$ is discrete and the singleton sets form a countable basis of clopen sets.

7 Pervin spaces on $\mathcal{P}(X)$

Let X be a set. For each subset L of X , let

$$\widetilde{L} = \{S \in \mathcal{P}(X) \mid S \cap L \neq \emptyset\}$$

Let $(X, \mathcal{V}_{\mathcal{L}})$ be a Pervin space. One can also make $\mathcal{P}(X)$ a Pervin space by considering the lattice $\widetilde{\mathcal{L}}$ of $\mathcal{P}(X)$ generated by the sets \widetilde{L} for each $L \in \mathcal{L}$. In other words, the subsets of $\mathcal{P}(X) \times \mathcal{P}(X)$ of the form

$$V_{\widetilde{L}} = \{(S, T) \in \mathcal{P}(X) \times \mathcal{P}(X) \mid (S \cap L \neq \emptyset \Rightarrow T \cap L \neq \emptyset)\},$$

where $L \in \mathcal{L}$, form a subbasis of the Pervin quasi-uniformity $\mathcal{V}_{\widetilde{\mathcal{L}}}$ on $\mathcal{P}(X)$.

Let now τ be a relation on $X \times Y$. If S is a subset of X , we set

$$\tau(S) = \{y \in Y \mid \text{there exists an } x \in S \text{ such that } (x, y) \in \tau\}$$

Since τ^{-1} is a relation on $Y \times X$, we also have, for each subset T of N

$$\tau^{-1}(T) = \{x \in X \mid \text{there exists an } y \in T \text{ such that } (x, y) \in \tau\}$$

One can also consider τ as a function $\widetilde{\tau}$ from X into $\mathcal{P}(Y)$. The interconnection between τ and $\widetilde{\tau}$ is easy to state.

Lemma 7.1 *For each $T \in \mathcal{P}(Y)$, one has $\widetilde{\tau}^{-1}(\widetilde{T}) = \tau^{-1}(T)$.*

Proof. One has, by definition,

$$\begin{aligned} \widetilde{\tau}^{-1}(\widetilde{T}) &= \{\widetilde{\tau}^{-1}(S) \mid S \in \widetilde{T}\} = \{\widetilde{\tau}^{-1}(S) \mid S \cap T \neq \emptyset\} \\ &= \{x \in X \mid \widetilde{\tau}(x) \cap T \neq \emptyset\} = \{x \in X \mid \tau(x) \cap T \neq \emptyset\} \\ &= \tau^{-1}(T) \quad \square \end{aligned}$$

Let $X [Y]$ be a Pervin space defined by a lattice $\mathcal{K} [\mathcal{L}]$ of subsets of $X [Y]$. We also consider $\mathcal{P}(Y)$ as a Pervin space equipped with the quasi-uniformity $\mathcal{V}_{\widetilde{\mathcal{L}}}$. The next result is now an easy consequence of Theorem 5.6 and Lemma 7.1.

Theorem 7.2 *Let τ be a relation on $X \times Y$. Then $\widetilde{\tau}$ is uniformly continuous if and only if, for every L in \mathcal{L} , $\tau^{-1}(L)$ belongs to \mathcal{K} .*

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