

Algebraic characterization of FO for scattered linear orderings

Alexis Bès¹ Olivier Carton²

¹LACL
Université Paris-Est

²LIAFA
Université Paris Diderot

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Monadic Second Order Logic (MSO)

Theorem (Büchi & others)

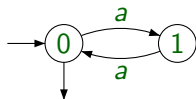
The following formalisms have the same expressive power for finite and infinite words

- ▶ *MSO*

$\exists X, Y \text{ partition}(X, Y) \wedge$

$\forall x \ x \in X \leftrightarrow x+1 \in Y \wedge \text{min} \in X \wedge \text{max} \in Y$

- ▶ *Automata*



- ▶ *Semigroups*

$\mathbb{Z}/2\mathbb{Z}$

- ▶ *Regular expressions*

$X = (aa)^*$

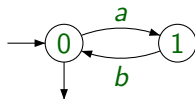


First Order Logic (FO)

Theorem (Schützenberger & others)

The following formalisms are equivalent for finite and infinite words

- ▶ FO $\forall x Q_a(x) \leftrightarrow Q_b(x+1) \wedge Q_a(\min) \wedge Q_b(\max)$
- ▶ Counter-free automata



- ▶ Aperiodic semigroups (satisfying $x^{\pi+1} = x^\pi$ for some π)

* $\beta\alpha$	β
α	* $\alpha\beta$

$$* 0$$

- ▶ Star-free expressions (with complementation)
 $X = (ab)^* = (a\bar{0} \cap \bar{0}b) \setminus (\bar{0}(aa + bb)\bar{0})$

Words

word: function from an ordering J (its **length**) to an alphabet A

Example $(a_j)_{j \in J}$	length J
$abaab$	5
$(ab)^\omega$	ω
$(ab)^{\tilde{\omega}}$	$\tilde{\omega}$
$(ab)^\omega c$	$\omega + 1$
$((ab)^\omega c)^\omega$	$\omega^2 = \omega \cdot \omega$
$(a^{\tilde{\omega}} b^\omega)^\omega$	$\zeta \cdot \omega$
a^{ω^ω}	ω^ω
$\begin{cases} r \mapsto a & \text{if } r = n/2^m \\ r \mapsto b & \text{if } r \in \mathbb{Q} \setminus \{n/2^m \mid m \geq 0\} \end{cases}$	η
$\begin{cases} x \mapsto a & \text{if } x \in \mathbb{Q} \\ x \mapsto b & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$	λ



Operations on orderings and words

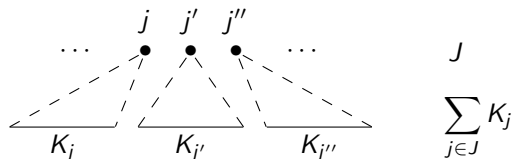
- ▶ Mirror image: \tilde{J} backwards linear ordering
- ▶ Sum of orderings and concatenation of words
 - ▶ $J + K$ places K to the right of J

$$\overline{\quad J \quad} \quad \overline{\quad K \quad}$$

- ▶ if x and y are of length J and K , xy is of length $J + K$.

- ▶ Generalized sum and concatenation

- ▶ $\sum_{j \in J} K_j$ generalized sum



- ▶ If each word x_j is of length K_j , $\prod_{j \in J} x_j$ is of length $\sum_{j \in J} K_j$



Scattered orderings

- ▶ **dense** ordering: $\forall i < j \exists k \quad i < k < j$.
- ▶ **scattered** ordering: no dense subordering.

Theorem (Hausdorff)

The class of countable scattered orderings is $\bigcup_{\alpha \in \mathcal{O}} V_\alpha$ where

- ▶ $V_0 = \{\mathbf{0}, \mathbf{1}\}$,
- ▶ $V_\alpha = \{\sum_{j \in J} K_j \mid J \in \{\omega, \tilde{\omega}\} \text{ and } K_j \in \bigcup_{\beta < \alpha} V_\beta\}$.

Example

$$\begin{aligned}(\omega + (\tilde{\omega})) \cdot \tilde{\omega} &= \cdots (\bullet \bullet \bullet \cdots \cdots \bullet \bullet \bullet) (\bullet \bullet \bullet \cdots \cdots \bullet \bullet \bullet) \\ &= \sum_{i \in \tilde{\omega}} \bullet \bullet \cdots \cdots \bullet \bullet = \sum_{i \in \tilde{\omega}} \left(\sum_{j \in \omega} \mathbf{1} + \sum_{j \in \tilde{\omega}} \mathbf{1} \right)\end{aligned}$$



Cuts of a linear ordering

A **cut** of a linear ordering J is a pair (K, L) of intervals such that

- ▶ $K \cup L = J$
- ▶ $\forall k \in K, \forall l \in L \quad k < l$.



\hat{J} linear ordering of the cuts

Examples

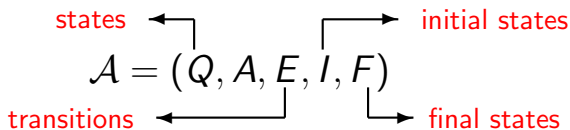
$$J = 3 \quad J \cup \hat{J} = | \bullet | \bullet | \bullet |$$

$$J = \omega \quad J \cup \hat{J} = | \bullet | \bullet | \bullet | \dots |$$

$$J = \zeta + \zeta \quad J \cup \hat{J} = | \dots | \bullet | \bullet | \bullet | \dots | \dots | \bullet | \bullet | \bullet | \dots |$$



Automata on linear orderings



$$E \subseteq Q \times A \times Q$$

successor transitions

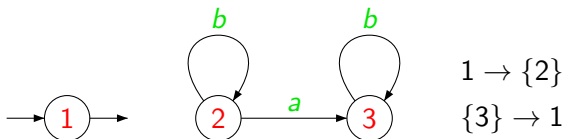
$$\cup \mathcal{P}(Q) \times Q$$

left limit transitions

$$\cup Q \times \mathcal{P}(Q)$$

right limit transitions

Example



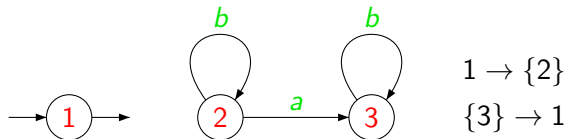
Paths

A **path** labeled by a word $x = (a_j)_{j \in J}$ is a sequence of states

$\gamma = (q_c)_{c \in \hat{J}}$ such that

- ▶ If $\begin{array}{c} | \\ a \\ | \\ p \quad q \end{array}$, then $p \xrightarrow{a} q \in E$,
- ▶ If $\begin{array}{c} \dots \\ | \\ P \quad q \end{array}$, then $P \rightarrow q \in E$,
- ▶ If $\begin{array}{c} | \\ \dots \\ q \quad P \end{array}$, then $q \rightarrow P \in E$.

Example



$$(b^{\tilde{\omega}} ab^{\omega})^2 \in L(\mathcal{A}) \quad \text{but} \quad (b^{\tilde{\omega}} ab^{\omega})^{\omega} \notin L(\mathcal{A})$$

$$\begin{array}{c} | \quad \dots b \quad | \quad b \quad | \quad a \quad | \quad b \quad | \quad b \dots \quad | \quad \dots b \quad | \quad b \quad | \quad a \quad | \quad b \quad | \quad b \dots \quad | \\ \hline 1 \quad \{2\} \quad 2 \quad 2 \quad 3 \quad 3 \quad \{3\} \quad 1 \quad \{2\} \quad 2 \quad 2 \quad 3 \quad 3 \quad \{3\} \quad 1 \end{array}$$



Monadic Second Order Logic (MSO)

Logic with

- ▶ order $<$ and unary predicates Q_a for each $a \in A$,
- ▶ boolean connectives \vee, \wedge, \neg ,
- ▶ quantifications on elements $\exists x$ and $\forall x$,
- ▶ quantifications on sets of elements $\exists X$ and $\forall X$.

Example

The following formula expresses that the ordering is infinite.

$$\exists X (\forall x \in X \exists y \in X \ x < y) \vee (\forall x \in X \exists y \in X \ x > y)$$

MSO on scattered orderings

Theorem (Bedon & others)

The following formalisms have the same expressive power for scattered and countable orderings

- ▶ *MSO*
- ▶ *Automata*
- ▶ *\diamond -Semigroups*
- ▶ *Regular expressions*

Remarks

- ▶ For **all** orderings, MSO is more expressive than automata
- ▶ One main ingredient is the closure under complementation.
- ▶ For the proof from automaton to formula, the usual trick to encode the run into a formula does not work. The formula quantifies over J whereas the run is over \hat{J} .



FO on cuts

Theorem (Bedon & Rispal)

The following formalisms have the same expressive power for *scattered and countable* orderings

- ▶ FO on cuts (First-order quantification over cuts)
 $\exists c \forall c' \exists c'' (c' < c \rightarrow c' < c'' < c) \wedge (c < c' \rightarrow c < c'' < c')$
- ▶ Aperiodic \diamond -semigroups (satisfying $x^{\pi+1} = x^\pi$ for some π)

* α	* α^ω
* $\alpha^{\tilde{\omega}}$	$\alpha^{\tilde{\omega}} \alpha^\omega$

$$* 0$$

- ▶ Star-free expressions
 $(A^\diamond(a + b)A^\diamond)^\omega (A^\diamond(a + b)A^\diamond)^{\tilde{\omega}}$



FO on elements

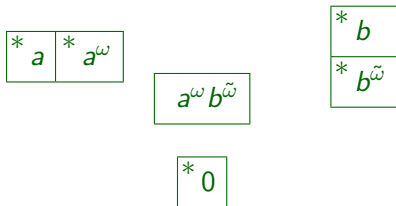
Theorem

The following formalisms have the same expressive power for *scattered and countable* orderings

- ▶ FO (First-order quantification over elements)

$$\begin{aligned} & \exists x Q_a(x) \wedge \exists y Q_b(y) \wedge \neg \exists x \exists y (x < y \wedge Q_b(x) \wedge Q_a(y)) \\ & \wedge \forall x (Q_a(x) \rightarrow \exists y (y > x \wedge Q_a(y))) \\ & \wedge \forall y (Q_b(y) \rightarrow \exists x (x < y \wedge Q_b(x))) \end{aligned}$$

- ▶ Aperiodic \diamond -semigroups satisfying $x^\omega x^{\tilde{\omega}} = x^\pi$ for some π



- ▶ Marked star-free expressions (Only products of the form XaY for some letter a)

$$(a^\diamond aa^\diamond)^\omega (b^\diamond bb^\diamond)^{\tilde{\omega}} = A^\diamond a A^\diamond b A^\diamond \setminus (A^\diamond b A^\diamond a A^\diamond \cup a^\diamond a b^\diamond \cup a^\diamond b b^\diamond)$$



The equation $x^\omega x^{\tilde{\omega}} = x^\pi$

It is well-known that FO can not distinguish the linear ordering $\omega + \tilde{\omega}$ from (large enough) finite linear orderings. Use Ehrenfeucht-Fraïssé games for instance.

This equation says that any word repeated ω times followed by the same word repeated $\tilde{\omega}$ times cannot be distinguish by FO from a (large enough) finite number of copies of this word.

This equation is sufficient to characterize FO on scattered linear orderings.

Summary

Logic	Algebraic object	Expressions
MSO	\diamond -semigroups	Regular Expressions
FO on cuts	aperiodic \diamond -semigroups	Starfree Expressions
FO on elements	\diamond -semigroups $x^\omega x^{\tilde{\omega}} = x^\pi$	Marked Starfree Expr.

Perspectives

- ▶ FO^2
- ▶ Countable linear orderings
- ▶ ...