

Alternation of quantifiers within FO^2

Pascal Weil

Joint work with Manfred Kufleitner

LaBRI, CNRS and Université de Bordeaux
IFMI, Universität Stuttgart

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- ▶ An analogy with the dot-depth hierarchy? Yes but no...

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- ▶ If r is a ranker, let $L(r)$ be the set of words on which r is defined
- ▶ (Schwentick, Thérien, Vollmer, 2001) A language is FO^2 -definable if and only if it is a Boolean combination of languages of the form $L(r)$.

Rankers: alternation

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Decidability? Algebraic characterization?

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Varieties of languages (at last!)

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- ▶ $\text{FO}_{m,n}^2$ and FO_m^2 are varieties of languages
- ▶ It is reasonable to look for an algebraic characterization

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- ▶ Let \mathbf{R}_m be the pseudovariety of monoids generated by the $A^*/\triangleright_{m,n}$ ($n \geq m$)
and let \mathbf{L}_m be the pseudovariety of monoids generated by the $A^*/\triangleleft_{m,n}$ ($n \geq m$)

Condensed rankers and deterministic products

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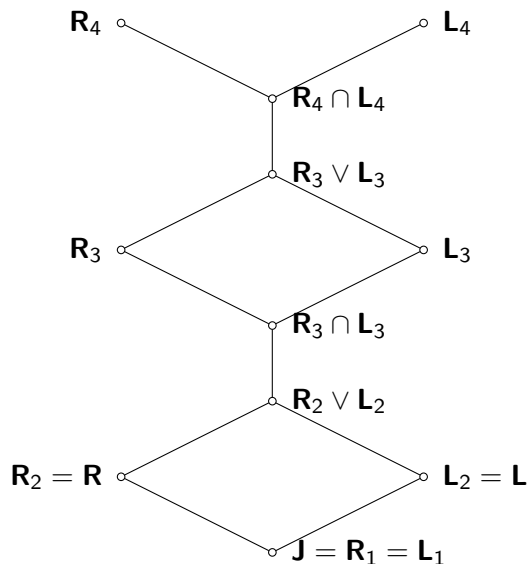
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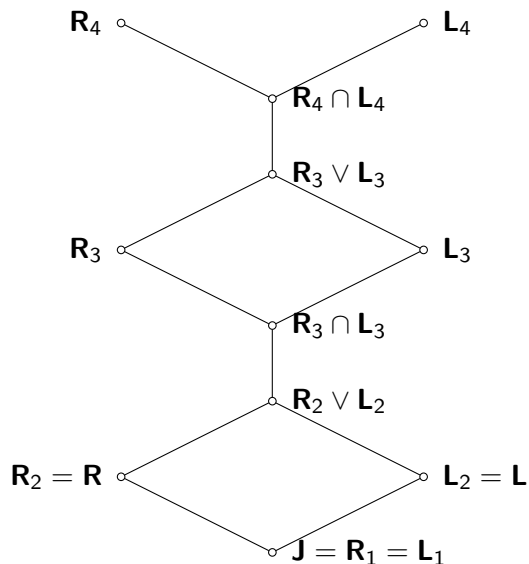
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- ▶ Right-left analogue: $\mathbf{L}_{m+1} = \mathbf{D} \textcircled{m} \mathbf{R}_m = \mathbf{R}_m^{co-det} = \mathbf{R}_m^{vco-det}$

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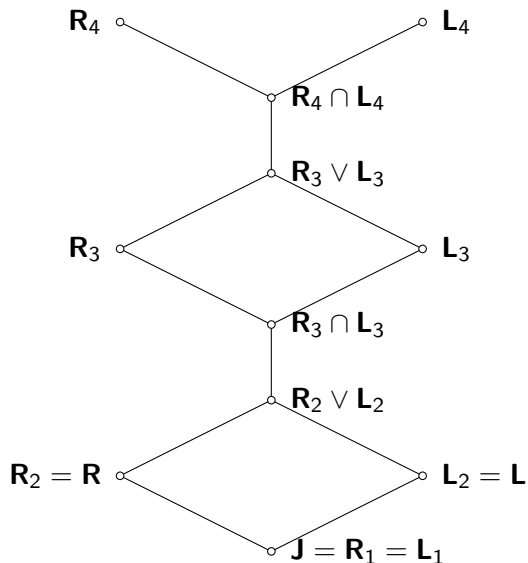


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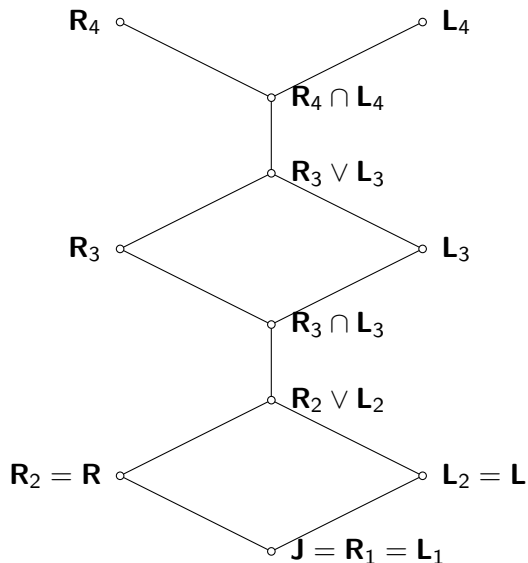
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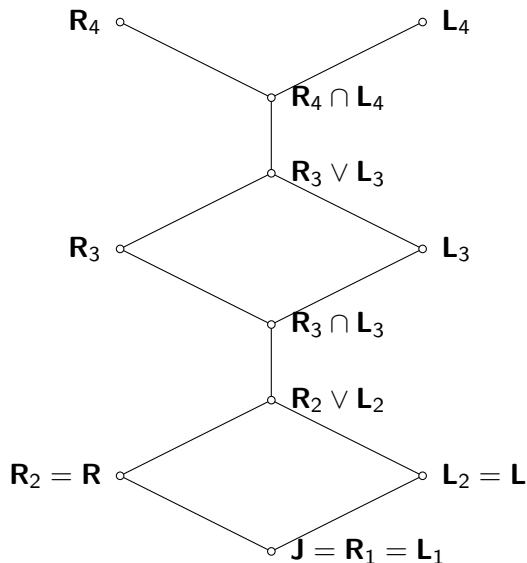


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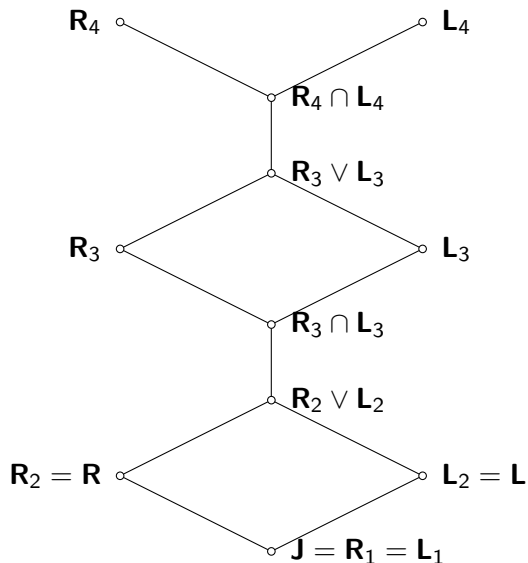
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Every m -generated element
of \mathbf{DA} is in $\mathbf{R}_{m+1} \cap \mathbf{L}_{m+1}$

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Theorem: u and v satisfy the same formulas in $FO_{m,n}^2$ (quantifier depth at most n , at most m blocks of alternating quantifiers) if and only if

- ▶ a ranker in $\underline{R}_{m,n}$ is **defined** on u iff it is **defined** on v
- ▶ if $r \in \underline{R}_{m,n}$ and $r' \in \underline{R}_{m-1,n-1}$ are **defined** on u and v , then $r(u) \leq r'(u)$ iff $r(v) \leq r'(v)$
- ▶ if $r \in \underline{R}_{m,n}$ and $r' \in \underline{R}_{m,n-1}$ are **defined** on u and v and end with different direction letters, then $r(u) \leq r'(u)$ iff $r(v) \leq r'(v)$

Back to FO^2 , with condensed rankers

Weis and Immermann's result also holds if we restrict our consideration to condensed rankers

Theorem: u and v satisfy the same formulas in $FO_{m,n}^2$ (quantifier depth at most n , at most m blocks of alternating quantifiers) if and only if

- ▶ a ranker in $\underline{R}_{m,n}$ is **condensed** on u iff it is **condensed** on v
- ▶ if $r \in \underline{R}_{m,n}$ and $r' \in \underline{R}_{m-1,n-1}$ are **condensed** on u and v , then $r(u) \leq r'(u)$ iff $r(v) \leq r'(v)$
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Main technical lemma: *if u and v agree on condensed rankers in $\underline{R}_{m,n}$, then they agree on rankers in $\underline{R}_{m,n}$*

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The quantifier alternation hierarchy within FO^2 vs. condensed rankers

► **Theorem.** $R_m \vee L_m \subseteq FO_m^2 \subseteq R_{m+1} \cap L_{m+1}$

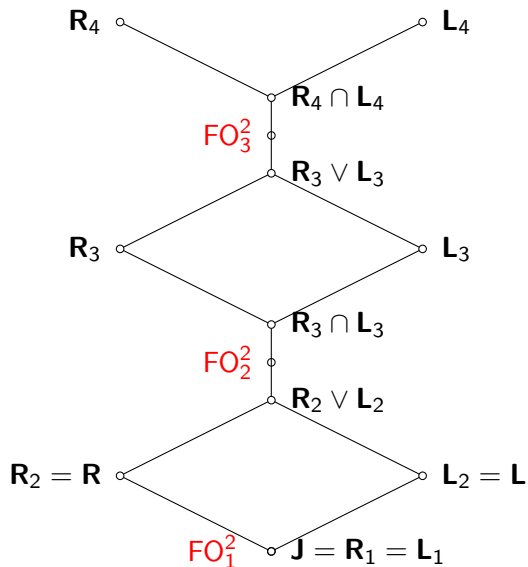
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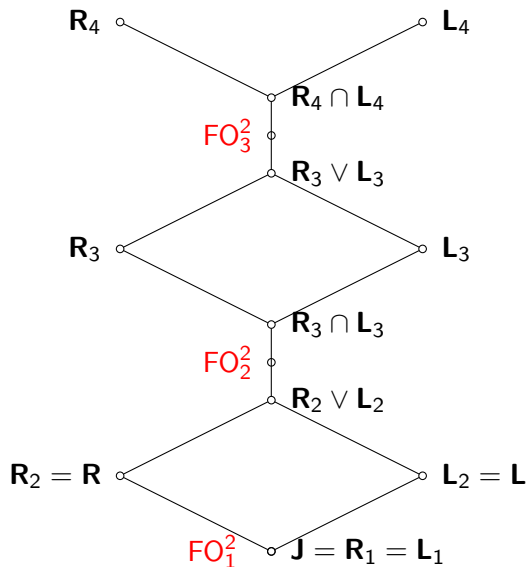
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- ▶ Technical result for the second containment: if $u \triangleright_{m+1, 2n} v$, then the condensed version of the conditions of Weis and Immerman hold for (m, n)

Where the quantifier alternation hierarchy fits



Where the quantifier alternation hierarchy fits



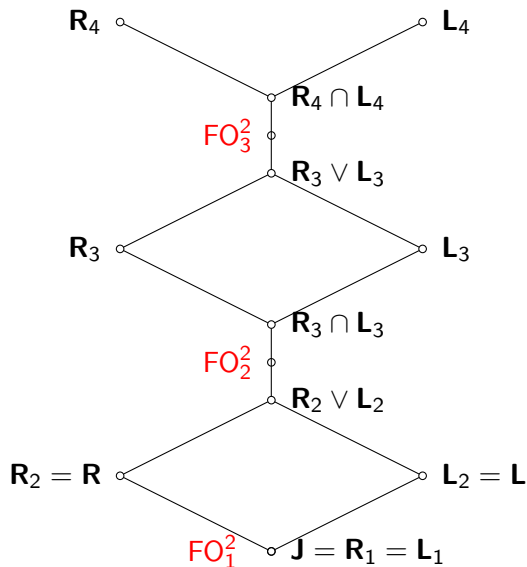
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Uses the identities of $\mathbf{R} \vee \mathbf{L}$

(Almeida, Azevedo, 1989)

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The FO_m^2 hierarchy
is decidable within one unit

- ▶ **Conjecture.** $\mathbf{R}_m \vee \mathbf{L}_m \subset \mathbf{FO}_m^2 = \mathbf{R}_{m+1} \cap \mathbf{L}_{m+1}$
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- ▶ Every unambiguous product of the form $B_0^* a_1 B_1^* \cdots a_k B_k^*$ (with $B_i \subseteq A$) can be expressed in terms of the B_i^* using only Boolean operations and at most $|A| + 1$ alternated applications of visibly deterministic and visibly co-deterministic products

Thank you for your attention!