

# A new and simple proof of Schützenberger's Theorem for star-free languages

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## Star-free and aperiodic languages

- ▶ **Star-free** languages  $SF(A^*)$ :
  - ▶  $A^* \in SF(A^*)$ .
  - ▶ For every letter  $a \in A$  we have  $\{a\} \in SF(A^*)$ .
  - ▶ If  $K, L \in SF(A^*)$ , then  $KL, K \cup L, K \setminus L \in SF(A^*)$ .
- ▶ A monoid  $M$  is **aperiodic** if there exists  $n \in \mathbb{N}$  such that

$$x^n = x^{n+1} \quad \text{for all } x \in M.$$

- ▶ **Aperiodic** languages  $AP(A^*)$ :  
 $L \in AP(A^*)$  if its syntactic monoid is finite and aperiodic.
- ▶ **Theorem (Schützenberger 1965)**:  $SF(A^*) = AP(A^*)$ .

## Simple properties

- ▶ Star-free languages are rational.
- ▶ Let  $B \subseteq A$ . Then  $B^* = A^* \setminus \bigcup_{b \notin B} A^* b A^*$ .
- ▶ If  $B \subseteq A$ , then  $\text{SF}(B^*) \subseteq \text{SF}(A^*)$ .
  
- ▶ **Lemma:** Let  $M$  be aperiodic. Then  $x_1 \cdots x_k = 1$  in  $M$  if and only if  $x_i = 1$  for all  $i$ .
- ▶ **Proof:** If  $xy = 1$ , then  $1 = xy = x^n y^n = x^{n+1} y^n = x$ . □

## A construction for monoids

- ▶ Let  $M$  be a monoid and let  $c \in M$ .
- ▶ Composition  $\circ$  on  $cM \cap Mc$  defined by  $xc \circ cy = xcy$ .
- ▶ Let  $xc = x'c$  and  $cy = cy'$ . Then

$$xc \circ cy = xcy = x'cy = x'cy' = x'c \circ cy'.$$

- ▶ Let  $cx = x'c$  and  $cy$  be elements in  $cM \cap Mc$ . Then

$$cx \circ cy = x'c \circ cy = x'cy = cxy.$$

- ▶ It follows
  - ▶  $(cM \cap Mc, \circ, c)$  is a monoid.
  - ▶ If  $M$  is aperiodic, then  $cM \cap Mc$  is aperiodic.
  - ▶ If  $M$  is aperiodic and  $c \neq 1$ , then  $|cM \cap Mc| < |M|$ .

## AP( $A^*$ ) $\subseteq$ SF( $A^*$ ), part 1

**Lemma:** If  $L \subseteq A^*$  is recognized by some finite aperiodic monoid  $M$ , then  $L \in \text{SF}(A^*)$ .

**Proof:**

- ▶  $\varphi : A^* \rightarrow M$  homomorphism recognizing  $L$ .
- ▶ The proof is by induction on  $(|M|, |A|)$ .
- ▶  $L = \bigcup_{p \in \varphi(L)} \varphi^{-1}(p)$ ; hence, show  $\varphi^{-1}(p) \in \text{SF}(A^*)$ .
- ▶  $\varphi^{-1}(1) = \{a \in A \mid \varphi(a) = 1\}^* \in \text{SF}(A^*)$ .
- ▶ Define  $\varphi_c = \varphi|_{(A \setminus \{c\})^*}$
- ▶ Let now  $p \neq 1$ .
- ▶ 
$$\varphi^{-1}(p) = \bigcup_{\substack{c \in A \\ \varphi(c) \neq 1}} \bigcup_{p=p_1 p_2 p_3} \varphi_c^{-1}(p_1) \cdot (\varphi^{-1}(p_2) \cap cA^* \cap A^*c) \cdot \varphi_c^{-1}(p_3)$$
- ▶  $\varphi_c^{-1}(p_i) \in \text{SF}((A \setminus \{c\})^*) \subseteq \text{SF}(A^*)$ .
- ▶ It remains to show  $\varphi^{-1}(p) \cap cA^* \cap A^*c \in \text{SF}(A^*)$ .

## $AP(A^*) \subseteq SF(A^*)$ , part 2

$\varphi(p) \cap cA^* \cap A^*c \in SF(A^*)$  for  $p \in \varphi(c)M \cap M\varphi(c)$ ,  $\varphi(c) \neq 1$ :

- ▶  $T = \varphi_c((A \setminus \{c\})^*) \leq M$
- ▶ Substitution: (here  $v_i \in (A \setminus \{c\})^*$ )  
 $\sigma : cA^* \rightarrow T^* : cv_1 \cdots cv_k \mapsto \varphi_c(v_1) \cdots \varphi_c(v_k)$
- ▶ Homomorphism:  
 $\psi : T^* \rightarrow \varphi(c)M \cap M\varphi(c)$  defined by  $\varphi_c(v) \mapsto \varphi(cvc)$ .
- ▶ Let  $w = cv_1 \cdots cv_k \in cA^*$ .
- ▶ We have  $wc \in \varphi^{-1}(p) \Leftrightarrow w \in \sigma^{-1}\psi^{-1}(p)$ ,  
i.e.,  $\varphi^{-1}(p) \cap cA^* \cap A^*c = \sigma^{-1}\psi^{-1}(p) \cdot c \cup \{c\}$ :

$$\begin{aligned}\psi\sigma(w) &= \psi(\varphi_c(v_1)\varphi_c(v_2) \cdots \varphi_c(v_k)) \\ &= \varphi(cv_1c) \circ \varphi(cv_2c) \circ \cdots \circ \varphi(cv_kc) \\ &= \varphi(cv_1cv_2 \cdots cv_kc) = \varphi(wc)\end{aligned}$$

- ▶  $\psi^{-1}(p) \in SF(T^*)$  and  $\varphi_c^{-1}(t) \in SF(A^*)$  for all  $t \in T$ .

## $AP(A^*) \subseteq SF(A^*)$ , part 3

- ▶ Remember  $\sigma : cA^* \rightarrow T^* : cv_1 \cdots cv_k \mapsto \varphi_c(v_1) \cdots \varphi_c(v_k)$ .
- ▶ For  $K, K_1, K_2 \in SF(T^*)$  we have

$$\sigma^{-1}(t) = c \cdot \varphi_c^{-1}(t) \quad \text{for } t \in T$$

$$\sigma^{-1}(T^* \setminus K) = cA^* \setminus \sigma^{-1}(K)$$

$$\sigma^{-1}(K_1 \cup K_2) = \sigma^{-1}(K_1) \cup \sigma^{-1}(K_2)$$

$$\sigma^{-1}(K_1 \cdot K_2) = \sigma^{-1}(K_1) \cdot \sigma^{-1}(K_2)$$

- ▶ Hence,  $\sigma^{-1}\psi^{-1}(p) \in SF(A^*)$ .



$$\text{SF}(A^*) \subseteq \text{AP}(A^*)$$

**Lemma:** For every  $L \in \text{SF}(A^*)$  there exists  $n(L) \in \mathbb{N}$  such that

$$xu^{n(L)}y \in L \Leftrightarrow xu^{n(L)+1}y \in L.$$

**Proof:**

- ▶  $n(\{a\}) = 2$  for  $a \in A$ .
- ▶  $n(A^* \setminus L) = n(L)$ .
- ▶  $n(K \cup L) = \max(n(K), n(L))$ .
- ▶  $n(KL) = n(K) + n(L)$ .





**Thank you!**