

Equations for the polynomial closure

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Classes of languages

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Variety of languages \mathcal{V} :

$$\begin{array}{ccc} A & \longmapsto & (A^*)\mathcal{V} \\ \text{alphabet} & & \text{subset of } \text{Rat}(A^*) \end{array}$$

such that

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- 2 $(A^*)\mathcal{V}$ is closed under quotients: $a^{-1}L, La^{-1} \in (A^*)\mathcal{V}$, for any $L \in (A^*)\mathcal{V}$.
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Lattice of languages of A^* : set of languages of A^* closed under finite union and finite intersection.

Theorem (Gehrke, Grigorieff, Pin)

A set \mathcal{L} of languages of A^ is a lattice of languages closed under quotients if and only if, for some set Σ of equations of the form $u \leq v$, with $u, v \in \widehat{A^*}$, \mathcal{L} is the set of the languages of A^* that satisfy all equations of Σ .*

Polynomial closure

The **polynomial closure** $\text{Pol}(\mathcal{L})$ of a set of languages \mathcal{L} of A^* is the set of languages that are **finite unions of marked products** of the form $L_0 a_1 L_1 \cdots a_n L_n$, where the a_i are letters and the L_i are in \mathcal{L} .

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Examples:

The **polynomial closure** of the set of languages of the form F or F^* , where F is a **finite** language is formed by the languages that have **star-height one**.

The **polynomial closure** of $\{\emptyset, A^*\}$ is formed by the **shuffle ideals** of A^* ($u \in L, v \in A^* \implies u \sqcup v \subseteq L$).

Polynomial closure

$$L_0, L_1, \dots, L_n \subseteq A^*, \quad a_1, \dots, a_n \in A$$

$$L = L_0 a_1 L_1 \cdots a_n L_n$$

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Syntactic morphisms:

$$\eta_i: A^* \rightarrow M(L_i), \quad \mu: A^* \rightarrow M(L)$$

Consider the morphism

$$\begin{aligned} \eta: A^* &\longrightarrow M(L_0) \times \cdots \times M(L_n) \\ u &\longmapsto (\eta_0(u), \dots, \eta_n(u)) \end{aligned}$$

and the relational morphism

$$\tau = \eta\mu^{-1}: M(L) \longrightarrow M(L_0) \times \cdots \times M(L_n)$$

Proposition (Pin and Weil)

For each idempotent $e \in M(L_0) \times \cdots \times M(L_n)$, the ordered semigroup $\tau^{-1}(e)$ satisfies the identity $x^\omega y x^\omega \leq x^\omega$.

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Theorem (Pin and Weil)

Let \mathbf{V} be an \mathbf{M} -variety and let \mathcal{V} be the corresponding variety of languages. Then $\text{Pol}(\mathcal{V})$ is a positive variety of languages and the corresponding \mathbf{OM} -variety is the Malcev product $\llbracket x^\omega y x^\omega \leq x^\omega \rrbracket \textcircled{\mathbf{M}} \mathbf{V}$.

Theorem (Pin and Weil)

$\llbracket x^\omega y x^\omega \leq x^\omega \rrbracket \textcircled{\mathbf{M}} \mathbf{V}$ is defined by the identities $x^\omega y x^\omega \leq x^\omega$, where $x, y \in \widehat{A}^*$ are such that the identity $y = x = x^2$ holds in \mathbf{V} .

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$\Sigma(\mathcal{L})$: the set of equations of the form $x^\omega y x^\omega \leq x^\omega$, where $x, y \in \widehat{A}^*$ are such that the equations $y \leq x$ and $x = x^2$ are satisfied by \mathcal{L} .

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Idea of proof: later

An example

A regular language is **slender** iff it is a finite union of languages of the form $u_0 v^* u_1$, where u_0, v, u_1 are words.

Regular **slender** or **full** languages of A^* form a lattice of closed under quotients \mathcal{S} .

The languages of $\text{Pol}(\mathcal{S})$ are finite unions of languages of the form $L_0 a_1 L_1 \cdots a_n L_n$, where the a_i are letters and the L_i are languages of the form A^* or u^* for some word u .

Theorem

A regular language of A^ belongs to $\text{Pol}(\mathcal{S})$ iff it satisfies the equations of the form*

$$(x^\omega y^\omega)^\omega z (x^\omega y^\omega)^\omega \leq (x^\omega y^\omega)^\omega$$

where $z \in A^$ and $x, y \in A^+$ and $i(x) \neq i(y)$.*

An example

Let $L \subseteq A^*$ regular.

$\eta: A^* \rightarrow M(L)$ syntactic morphism.

By last theorem, $L \in \text{Pol}(\mathcal{S})$ iff for any $z \in A^*$ and $x, y \in A^+$ s.t.
 $i(x) \neq i(y)$,

$$(\hat{\eta}(x)^\omega \hat{\eta}(y)^\omega)^\omega \hat{\eta}(z) (\hat{\eta}(x)^\omega \hat{\eta}(y)^\omega)^\omega \leq (\hat{\eta}(x)^\omega \hat{\eta}(y)^\omega)^\omega.$$

Let

$$F = \bigcup_{a, b \in A, a \neq b} \eta(a)M(L) \times M(L)\eta(b)$$

Then

$L \in \text{Pol}(\mathcal{S})$ iff the property $(s^\omega t^\omega)^\omega r (s^\omega t^\omega)^\omega \leq (s^\omega t^\omega)^\omega$ holds for all
 $(s, t) \in F, r \in M(L)$.

An example

Thus we may conclude:

Proposition

There is an algorithm to decide whether a given regular language belongs to $Pol(S)$.

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Easier part

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Recall:

$\Sigma(\mathcal{L})$: the set of equations of the form $x^\omega y x^\omega \leq x^\omega$, where $x, y \in \widehat{A}^*$ are such that the equations $y \leq x$ and $x = x^2$ are satisfied by \mathcal{L} .

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$L \subseteq A^*$ regular.

Define

$$E_L = \left\{ (x, y) \in \widehat{A}^* \times \widehat{A}^* \mid L \text{ satisfies } y \leq x \text{ and } x = x^2 \right\}$$

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E_L and F_L are clopen in $\widehat{A}^* \times \widehat{A}^*$.

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