λ-calculus and recognizability

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Réunion de démarrage ANR FREC

Outline

Simply typed λ-calculus, free algebra and free monoid

Recognizability in the simply typed λ-calculus

A machine-like characterization of recognizability

Closure properties

Some applications

Perspectives within FREC

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Simply typed λ-calculus

Given a finite set of atomic types $\mathcal{A}$, simple types are:

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A higher order signature is a tuple $\Sigma = (\mathcal{A}, \mathcal{C}, \tau)$ where:

- $\mathcal{A}$ is a finite set of atomic types,
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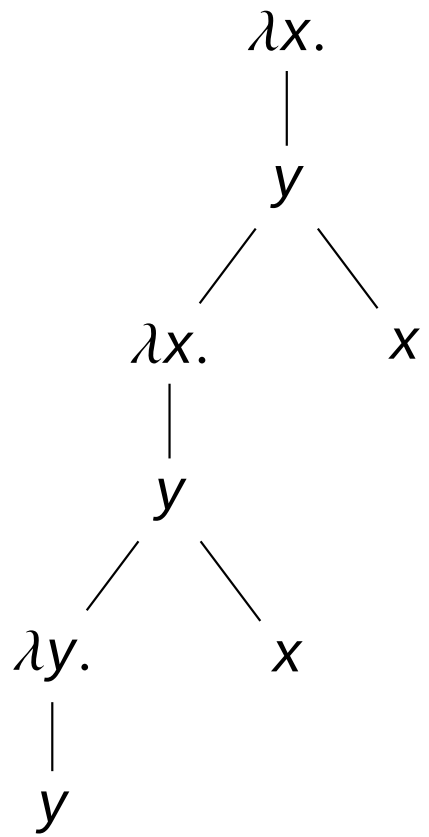
λ-terms built on Σ are defined as:

- for $\alpha \in \mathcal{T}_{\mathcal{A}}$, $x^{\alpha} \in \Lambda_{\Sigma}^{\alpha}$,
- $c \in \Lambda_{\Sigma}^{\tau(c)}$,
- if $M_1 \in \Lambda_{\Sigma}^{\alpha_2 \multimap \alpha_1}$, $M_2 \in \Lambda_{\Sigma}^{\alpha_2}$, then $(M_1 M_2) \in \Lambda_{\Sigma}^{\alpha_1}$,
- if $M \in \Lambda_{\Sigma}^{\alpha_1}$, then $\lambda x^{\alpha_2}.M \in \Lambda_{\Sigma}^{\alpha_2 \to \alpha_1}$.

Free and bound variables



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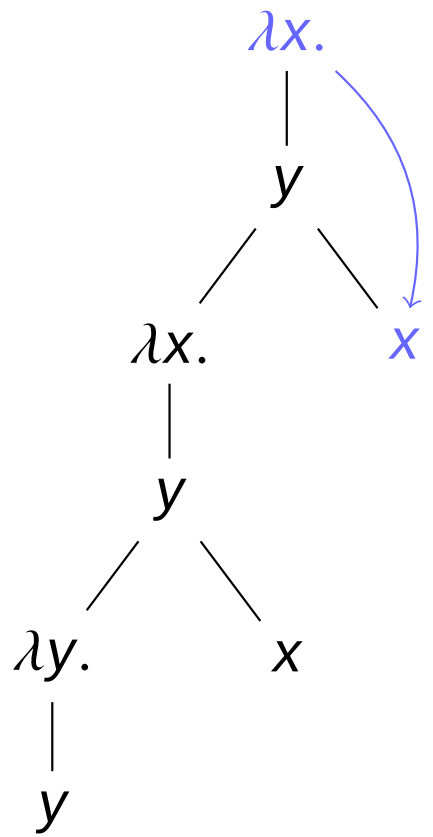

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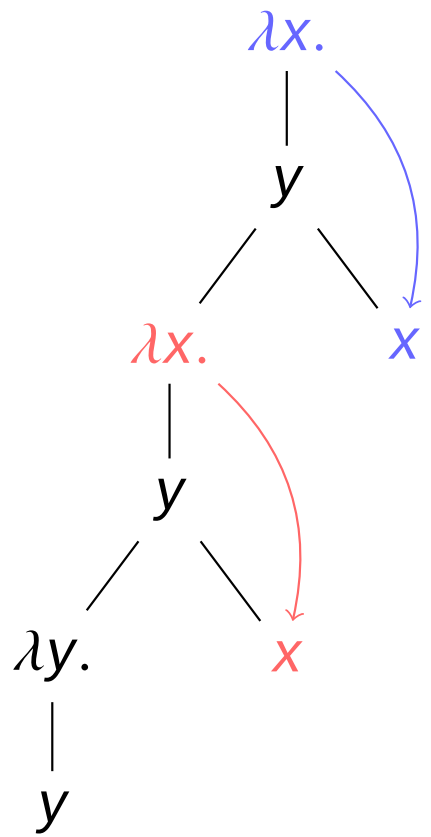

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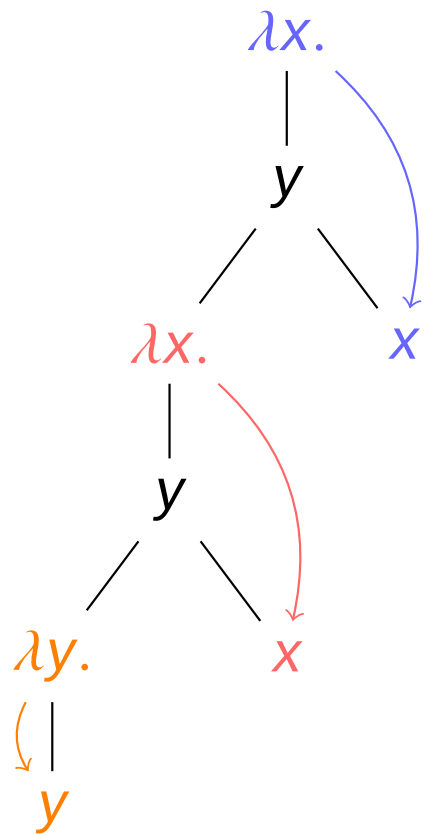

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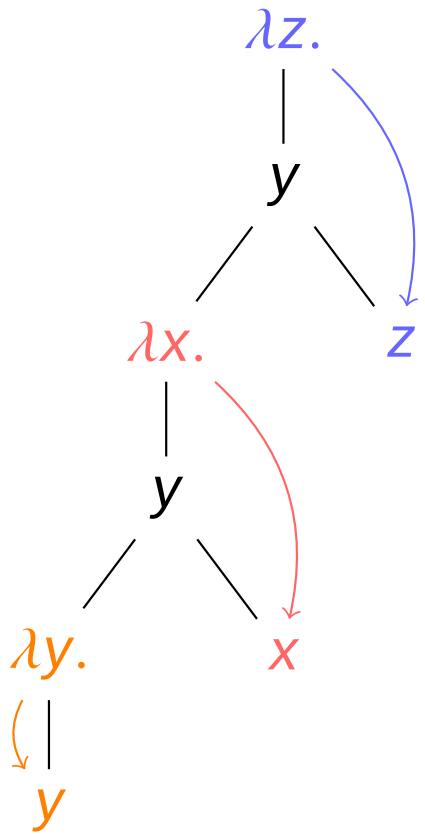

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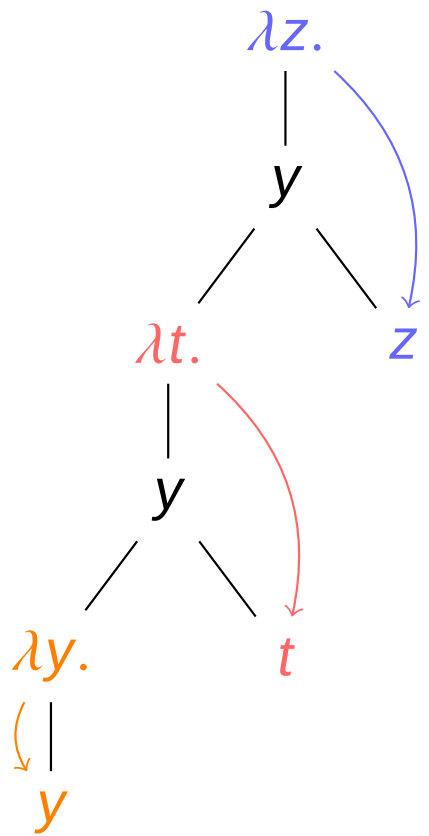

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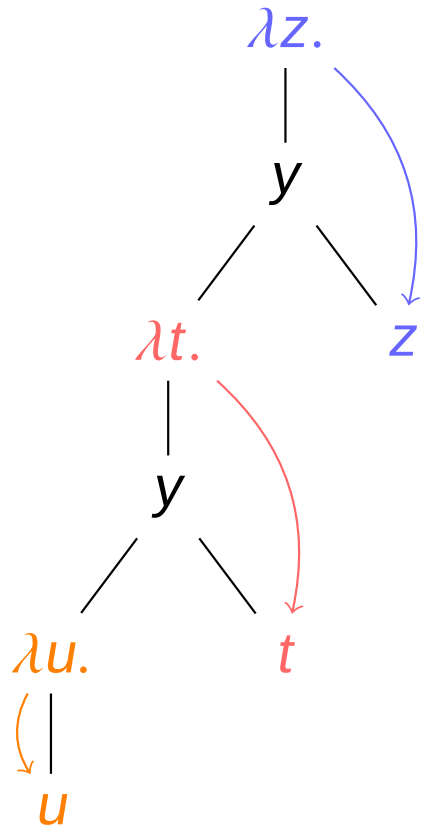

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- $\forall : (e \to t) \to t, \exists : (e \to t) \to t$
- $love : e \to e \to t$,$woman : e \to t$, $man : e \to t$, $mary : e$, $john : e \ldots$

$$\forall(\lambda x^e.\exists(\lambda y^e. \wedge (man\, x^e)(\wedge(woman\, y^e)(love\, x^e\, y^e)))$$

$$\approx$$

$$\forall x.\exists y.man(x) \wedge woman(y) \wedge love(x, y)$$

Operational semantics: β-reduction

λ-calculus is a theory of function and computation.
Computation is done with the relation of β-contraction ($\rightarrow_\beta$):

$$\frac{(\lambda x.M)N}{(\lambda x.M)N \rightarrow_\beta M[x := N]} \qquad \frac{M_1 \rightarrow_\beta M_2}{(MM_1) \rightarrow_\beta (MM_2)}$$

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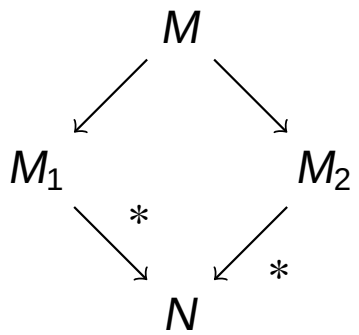
β-reduction ($\overset{*}{\rightarrow}_\beta$): reflexive transitive closure of β-contraction

Theorem (Church-Rosser)



Operational semantics:η-reduction

Mathematicians do not make a difference between the function sin and the function $f : x \to \sin(x)$.
η-contraction is used to reflect this fact in the λ-calculus.

$$\frac{(\lambda x.Mx) \quad x \notin FV(M)}{M} \qquad \frac{M_1 \to_\eta M_2}{(MM_1) \to_\eta (MM_2)}$$

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Church-Rosser Theorem holds for all those relations

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- No matter how a term is reduced, it will always output a result, and the same result.

Simply typed λ-calculus generalizes trees

The ranked alphabet $\{e; g; f\}$ where $\mathrm{rank}(e) = 0$, $\mathrm{rank}(g) = 1$, $\mathrm{rank}(f) = 2$ can be represented by the following second order constants:

$$e : o,\ g : o \to o,\ f : o \to o \to o$$

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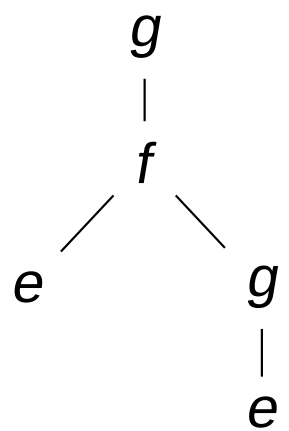
The Böhm tree of the λ-term is the same as the graphic representation of the term:



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Concatenation is then $s_1 + s_2 = \lambda x^o.s_1(s_2(x^o))$:

$$\begin{array}{rcl} /ab/ + /bb/ & = & \lambda x^o.a(b(x^o)) + \lambda x^o.b(b(x^o)) \\ & = & \lambda x^o.(\lambda y^o.a(b\,y^o))((\lambda z^o.b(b\,z^o))x^o) \\ & =_{\beta\eta} & \lambda x^o.a(b(b(b\,z^o))) \end{array}$$

and the empty string is $\lambda x^o.x^o$

Formal language theory and λ-calculus

- tree generated by a ranked alphabet: free algebra
- string generated by an alphabet: free monoid
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Languages of λ-terms:

- Abstract Categorial Grammars
- Sets of semantic representations in Montague semantics
- Solutions of higher-order matching equations
- Finer grained languages than those obtained in substructural logic may be necessary in linguistics for: traces, extractions, ellipsis...

Recognizability

- For strings and trees, it corresponds to one of the simplest classes of languages,
- It naturally has several characterisation:
  - finite state automaton,
  - regular expressions,
  - finite congruences (Myhill-Nerode),
  - logic (W1S, WkS) (Büchi, Thatcher and Right, Döner),
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We here present a generalization of recognizability for trees and strings to the simply typed λ-calculus.

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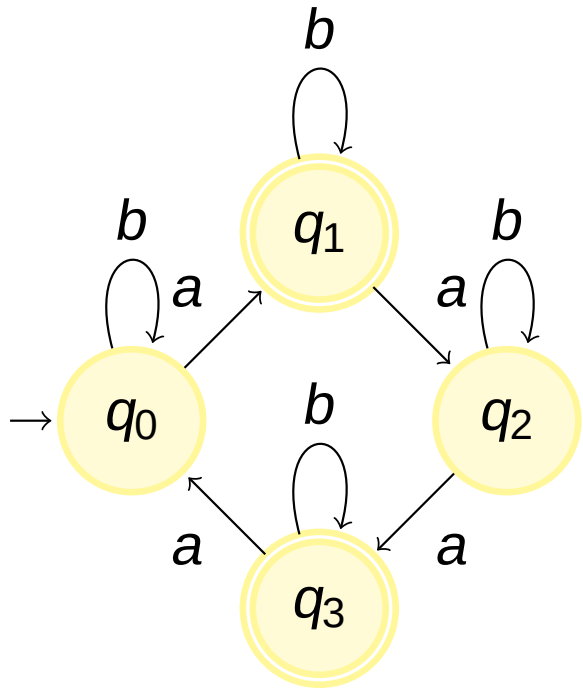
String recognizability and finite semigroup

Let's consider a deterministic total finite state automaton:



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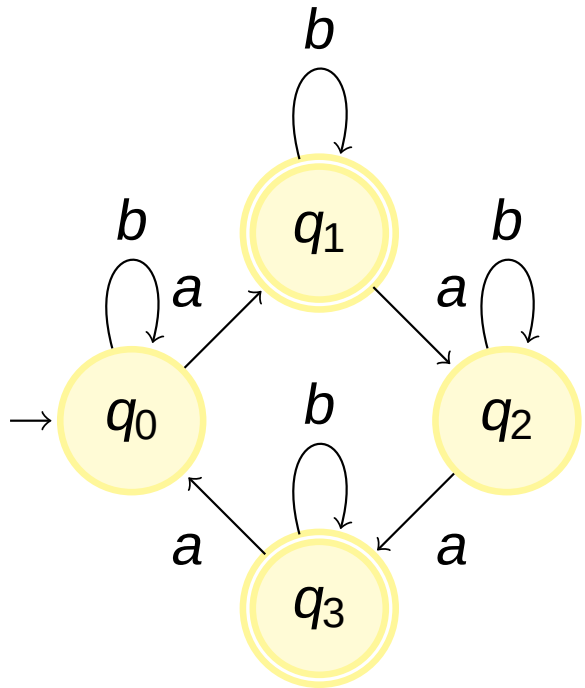

- Each letter, a or b, can be interpreted as a function f_a, f_b, from states to states such that $f_\alpha(q) = q'$ iff (q, α, q') is a transition.

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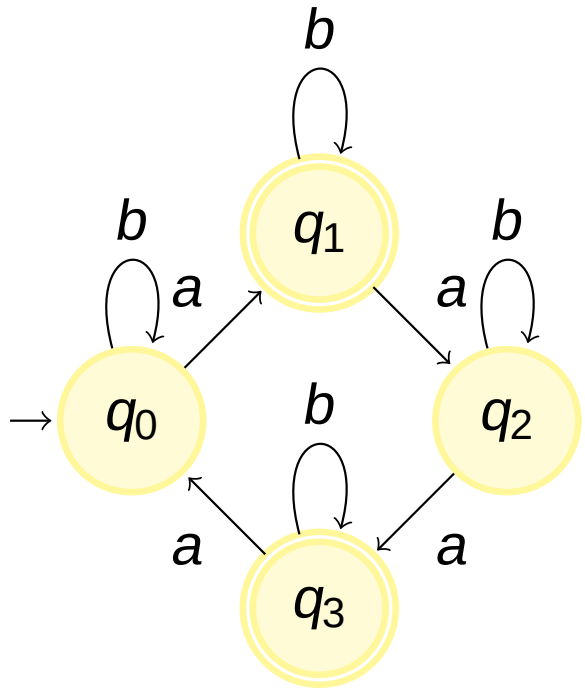

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- A word $\alpha_1 \ldots \alpha_n$ is in the language iff $h_{\alpha_1} \circ \ldots \circ h_{\alpha_n}(q_0) \in \{q_1; q_3\}$.

Tree recognizability and finite algebra

Let's consider the deterministic tree automata $\mathcal{A}$ with the states q_0, q_1, q_2 whose final states are $\{q_0; q_1\}$ and the rules:

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Then the automaton is computing the operations in the finite algebra $\mathbb{A}$, $\{q_0; q_1; q_2; \bot\}$ where $f(q_i, q_j) =_{\mathbb{A}} q_0$, $f(q, q') =_{\mathbb{A}} \bot$ when $q = \bot$ or $q' = \bot \ldots$ and

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Finite algebras and finite semi-groups are particular cases of finite models of the simply typed λ-calculus.

Finite models for recognizability in the simply typed λ-calculus

Let Σ be a signature. $\mathbb{M} = ((\mathcal{M}^\alpha)_{\alpha \in \mathcal{T}(\Sigma)}, \iota)$ is a finite model of Σ if:

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- $\mathcal{M}^{\alpha \to \beta}$ is the set of all functions from $\mathcal{M}^\alpha$ to $\mathcal{M}^\beta$.
- ι maps constants of type α to $\mathcal{M}^\alpha$

A variable assignment $\chi : V \to \bigcup_{\alpha \in \mathcal{T}(\Sigma)} \mathcal{M}^\alpha$ so that $\chi(x^\alpha) \in \mathcal{M}^\alpha$.
The semantics of λ-terms in $\mathbb{M}$ is inductively defined by:

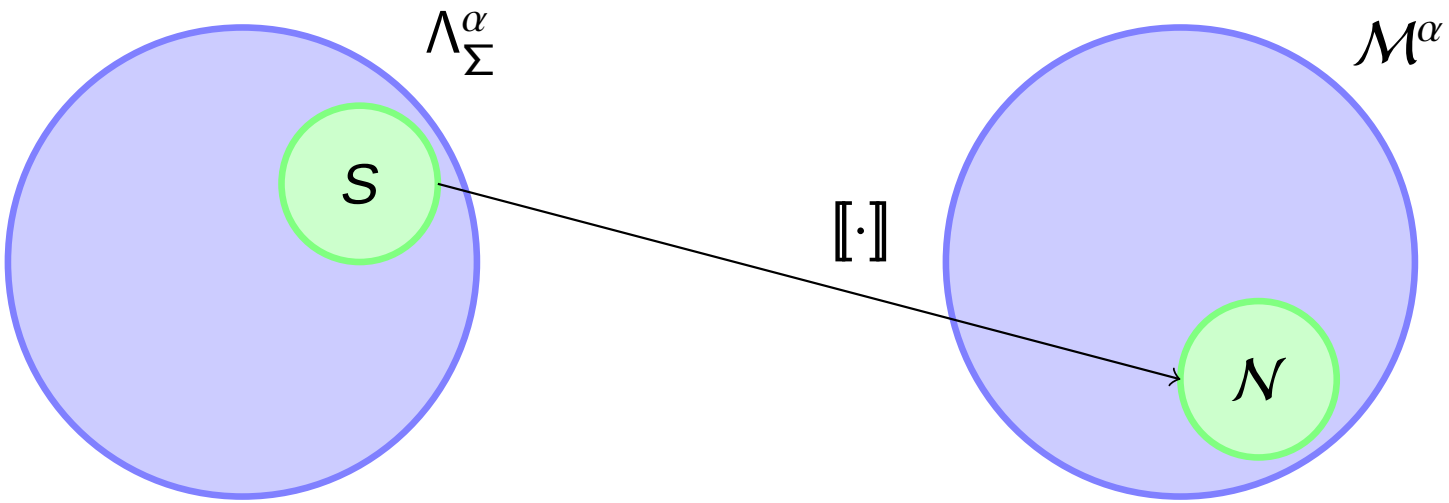
- $[\![c]\!]^{\mathbb{M}}_\chi = \iota(c)$,
- $[\![x^\alpha]\!]^{\mathbb{M}}_\chi = \chi(x^\alpha)$,
- $[\![MN]\!]^{\mathbb{M}}_\chi = [\![M]\!]^{\mathbb{M}}_\chi([\![N]\!]^{\mathbb{M}}_\chi)$,
- $[\![\lambda x^\alpha.M]\!]^{\mathbb{M}}_\chi = a \in \mathcal{M}^\alpha \to [\![M]\!]^{\mathbb{M}}_{\chi \leftarrow [x^\alpha := a]}$

Finite models for recognizability in the simply typed λ-calculus

Definition:

A set of λ-terms $S \subseteq \Lambda^{\alpha}_{\Sigma}$ is recognizable iff there is a finite set of variables W, a finite full model $\mathbb{M} = ((\mathcal{M}^{\alpha})_{\alpha \in \mathcal{T}(\Sigma)}, \iota)$, $\mathcal{N} \subseteq \mathcal{M}^{\alpha}$ and a variable assignment χ such that:

$$S = \{M | FV(M) \subseteq W \wedge [\![M]\!]^{\mathbb{M}}_{\chi} \in \mathcal{N}\}$$



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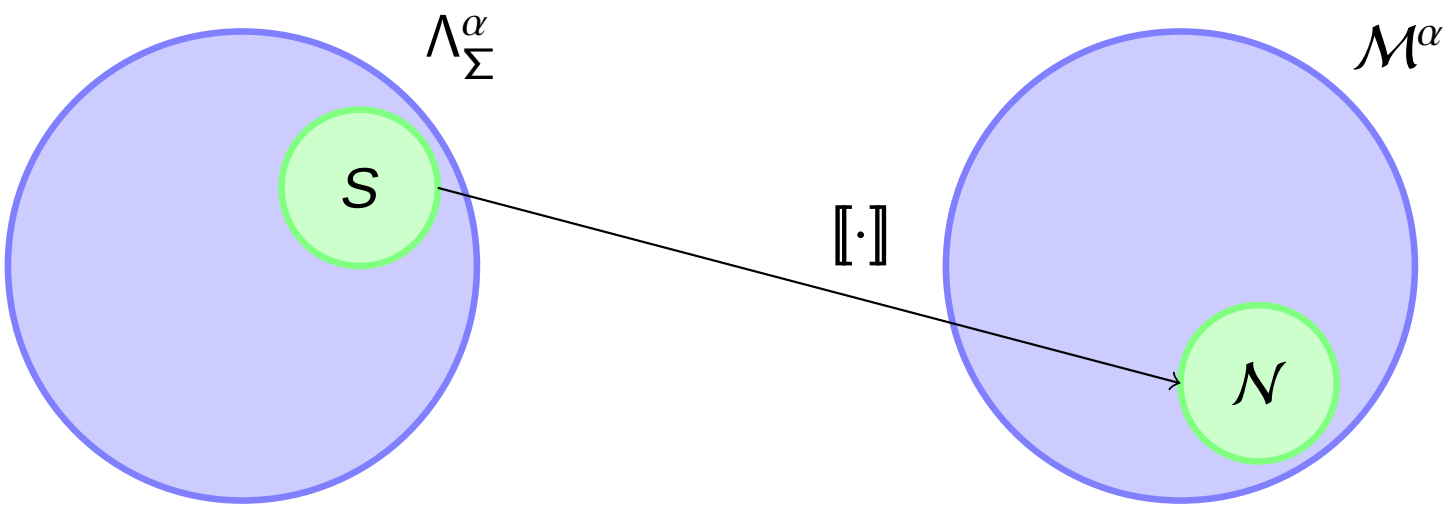

Note:

- recognizable sets are closed under $=_{\beta\eta}$
- the emptiness of recognizable sets subsumes λ-definability which is undecidable (Loader 1993).

An example: Quantified Boolean Formulae (QBF)

We use the signature with the constants: $\wedge : p \to p \to p$, $\vee : p \to p \to p$, $\neg : p \to p$, $\forall^2 : (p \to p) \to p$, $\exists^2 : (p \to p) \to p$
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We take the model $\mathbb{B} = ((\mathcal{B}^\alpha)_{\alpha \in \mathcal{T}(\Sigma)}, \iota)$ such that:

- $\mathcal{B}^p = \{0, 1\}$,
- $\wedge$, $\vee$, $\neg$ are interpreted as usual by ι
- $\iota(\forall^2)(f)$ is equal to 1 iff $f(1) = f(0) = 1$,
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Thus the set of closed terms that represent QBF tautologies is recognizable.

Outline

Obtaining recognizable sets of trees with types

Recognizability is invariant under $=_{\beta\eta}$, and the best way to capture such invariants in the λ-calculus is to use types.

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It is as if a should have at the same time the type $q_1^1 \to \cdots \to q_n^1 \to q^1, \ldots,$ and the type $q_1^p \to \cdots \to q_n^p \to q^p$.

Intersection types

There is a type system, intersection types (Coppo, Dezani 1980), that assignes several types to λ-terms:

- it is used to type the untyped λ-caculus and grasp dynamic properties of terms via typing (strong/weak normalization, solvability...),

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- typing judgements are closed under $=_{\beta\eta}$.

Intersection types a brief introduction

In the untyped λ-calculus, intersection type systems are of the form:

$$\frac{}{\Gamma, x : A \vdash x : A} \qquad \frac{}{\Gamma \vdash M : \Omega} \qquad \frac{\Gamma \vdash M : A \quad \Gamma \vdash M : B}{\Gamma \vdash M : A \cap B}$$

$$\frac{\Gamma \vdash M : A_1 \cap A_2 \quad i \in \{1; 2\}}{\Gamma \vdash M : A_i}$$

$$\frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B} \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B}$$

Let $\mathbf{S}(\Gamma, A) = \{M | \Gamma \vdash M : A\}$, one can easily see that:

- $\mathbf{S}(\Gamma, A \cap A) = \mathbf{S}(\Gamma, A)$ and
  $\mathbf{S}(\Gamma, \Omega \cap A) = \mathbf{S}(\Gamma, A \cap \Omega) = \mathbf{S}(\Gamma, A)$,
- $\mathbf{S}(\Gamma, A \cap B) = \mathbf{S}(\Gamma, B \cap A)$ and
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  - $\cap_\Pi^{\alpha\to\beta} = 2^{\cap_\Pi^\alpha} \times \{\alpha\} \times \cap_\Pi^\beta$.

Given $(S, \alpha, p) \in \cap_\Pi^{\alpha\to\beta}$, it may be written:

- $p_1 \cap \ldots \cap p_n \to p$ when $S \neq \emptyset$ and $S = \{p_1; \ldots; p_n\}$.
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N.B.: $|\cap_\Pi^\alpha|$ is finite.

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In general we write $\Gamma \vdash_\Pi M : S$ instead of $\forall p \in S.\, \Gamma \vdash_\Pi M : S$.

Subsumption relation on types

$$\frac{\iota \in \rho(\alpha)}{\iota \sqsubseteq^{\alpha}_{\Pi} \iota} \qquad \frac{S \subseteq \cap^{\alpha}_{\Pi} \quad \forall p \in T. \exists q \in S.\ q \sqsubseteq^{\alpha}_{\Pi} p}{S \trianglelefteq^{\alpha}_{\Pi} T}$$

$$\frac{S \trianglelefteq^{\alpha}_{\Pi} T \quad q \sqsubseteq^{\beta}_{\Pi} p}{(T, \alpha, q) \sqsubseteq^{\alpha \to \beta}_{\Pi} (S, \alpha, p)}$$

Properties

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- **Subject conversion:**
  $M,\ N \in \Lambda^\alpha_\Sigma \wedge M =_{\beta\eta} N \Rightarrow \Gamma \vdash_\Pi N : p$ iff $\Gamma \vdash_\Pi M : p$
- **Singleton set:** for every $M \in \Lambda^\alpha_\Sigma$ there are Π, Γ and p such that $\Gamma \vdash_\Pi N : p$ iff $N \in \Lambda^\alpha_\Sigma$ and $M =_{\beta\eta} N$.

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- extensions of the coherence Theorems such as [Aoto, Ono 99] lead to a different proof of the singleton theorem using shorter types,
- this theorem is tightly related to the finite completeness Theorem by [Statman 82]

Example of the singleton construction

The unique term that can be typed with $(0 \to 1 \cap 1 \to 2) \to 0 \to 2$ is $\lambda f^{*\to *} x^*.f^{*\to *}(f^{*\to *}(x^*))$:

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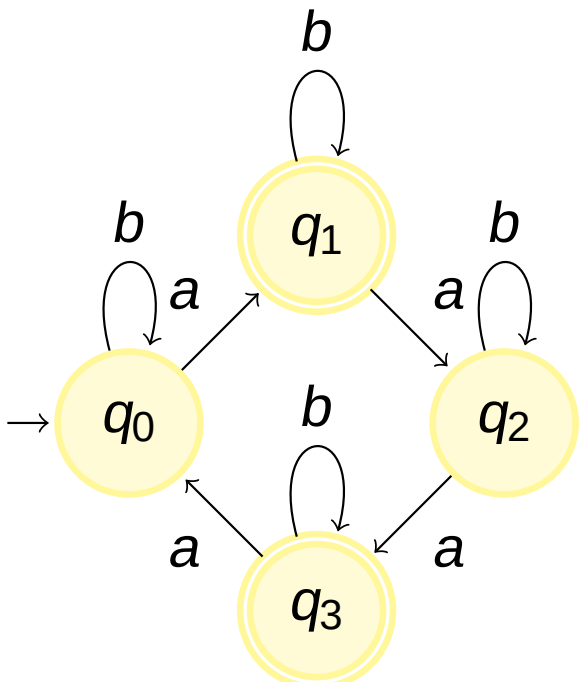

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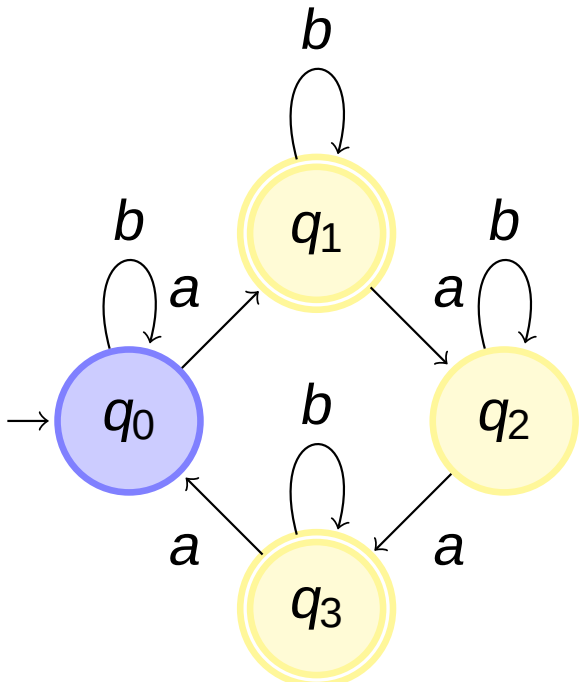

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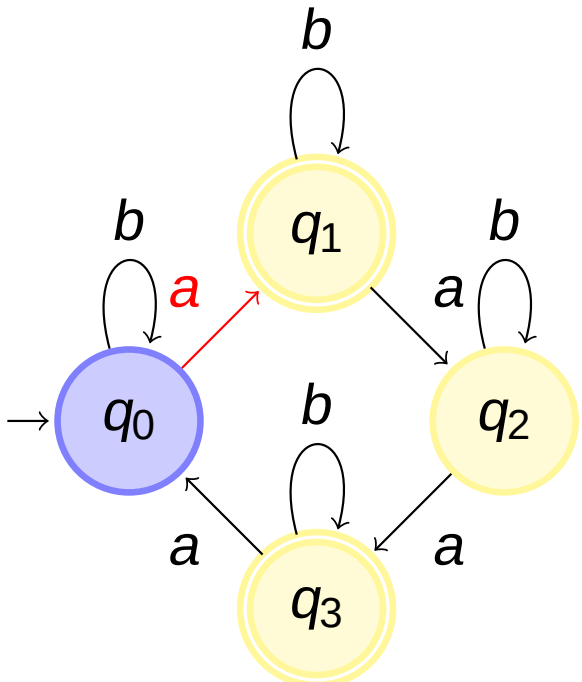

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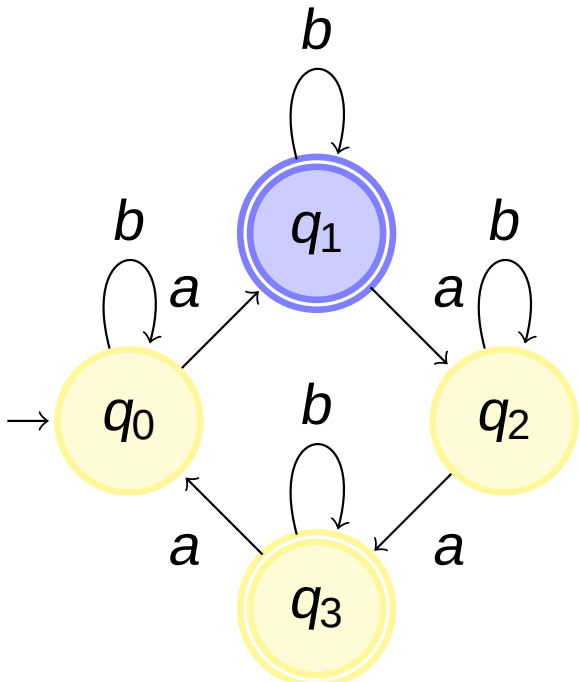

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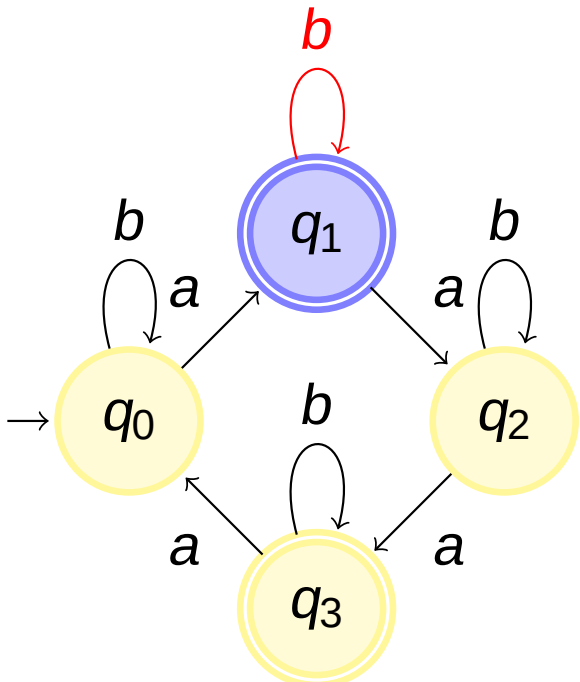

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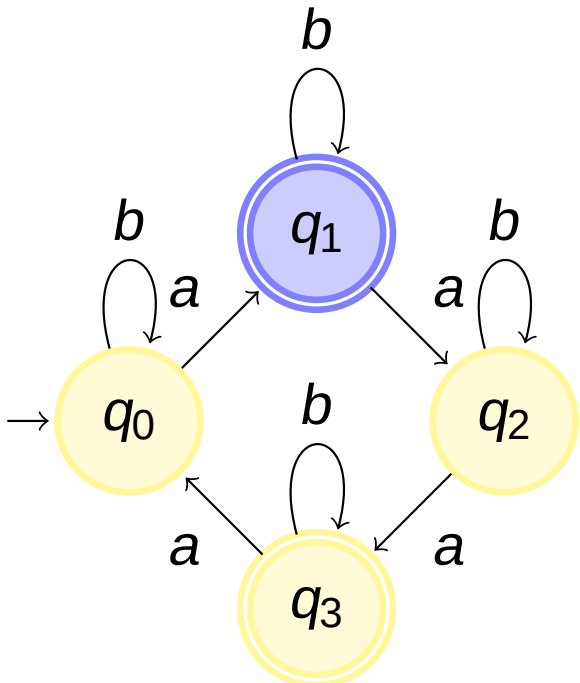

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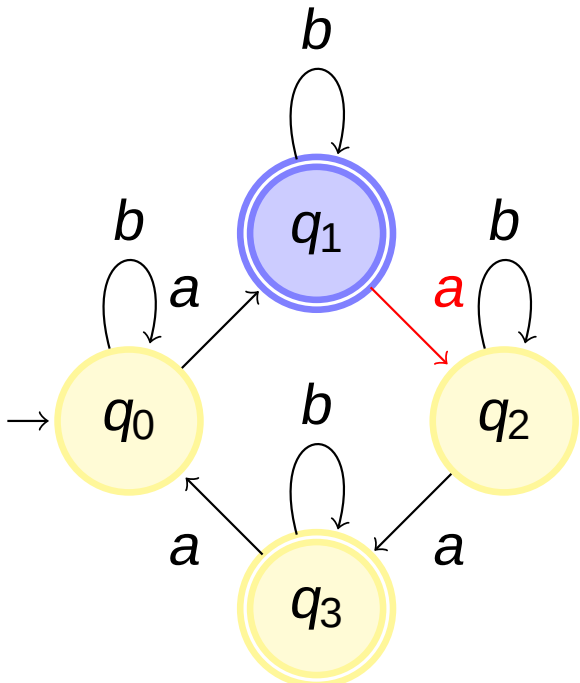

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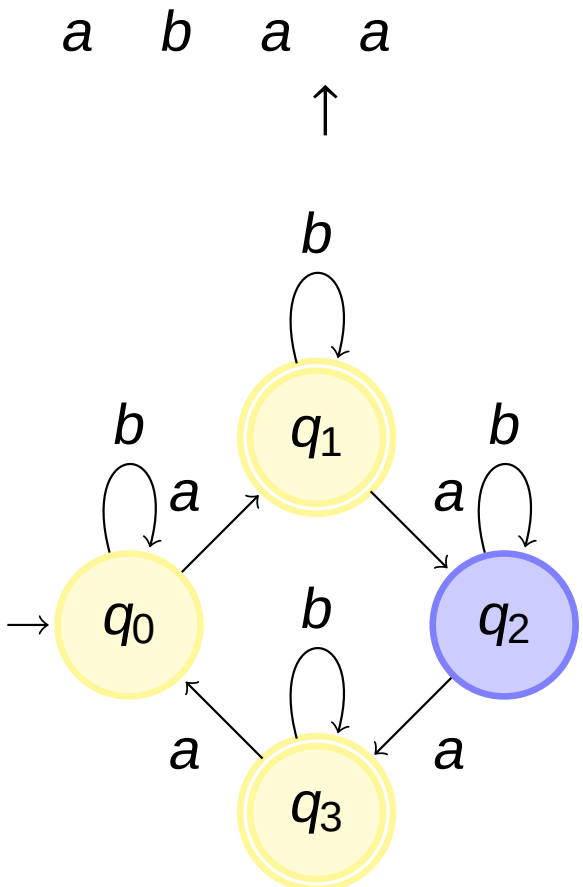

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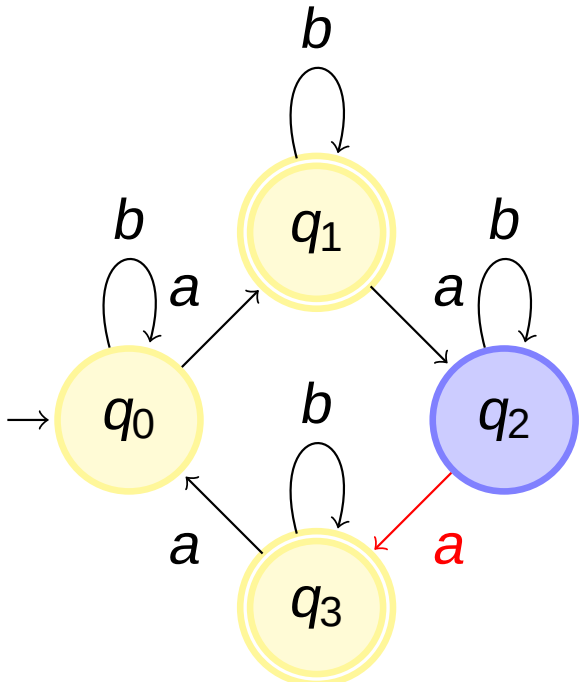

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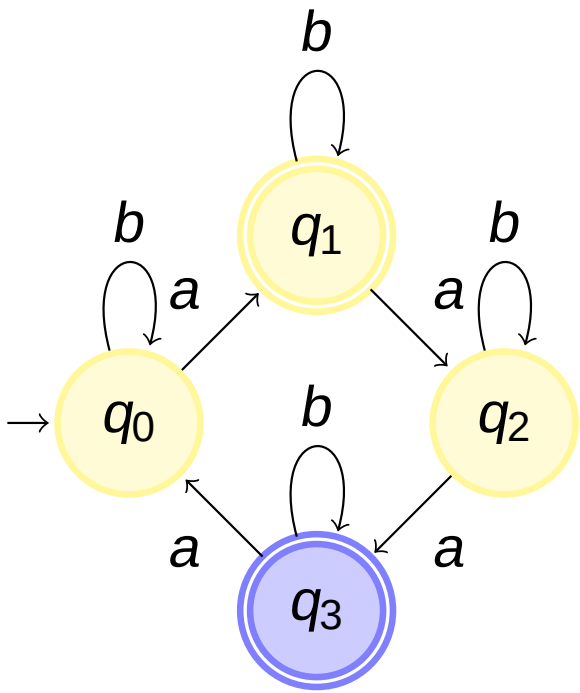

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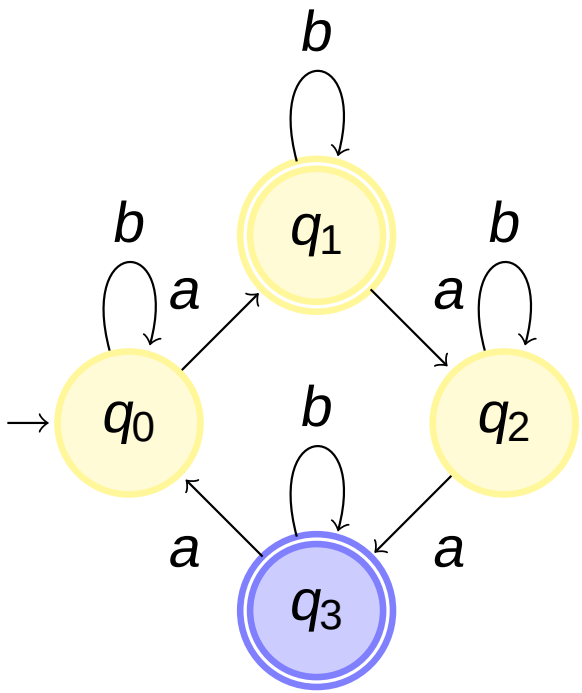

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HOIS and finite models

We have the two following effective theorems:

- **Theorem:** Given a model $\mathbb{M}$ and h in $\mathbb{M}$ there is a HOIS Π and a context Γ and a set of intersection types P such that $[\![M]\!]^{\mathbb{M}}_{\rho} = h$ iff the sequent $\Gamma \vdash_{\Pi} M : P$ is derivable.

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Note: the first Theorem can be use to reduce the emptiness of intersection types to λ-definability and give a simple proof of the undecidability result by [Urzyczyn 1999].

Intersection types as automata

The two previous theorems imply that S is a recognizable set of λ-terms iff there is a HOIS Π, a context Γ and $\{S_1; \ldots; S_n\}$ such that for every $M \in S$ there is $k \in [1; n]$ so that $\Gamma \vdash_\Pi M : S_k$ is derivable.

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Type-automata and finite state automata in a nutshell:

TA	**FSA**
types	states
typing axioms and typing rules	transitions
finite model	finite algebra/semigroup

Outline

Boolean closure properties

We can generalize the traditional constructions using automata to type-automata:

- intersection and union closure can be obtained by taking a generalization of the product of type-automata,

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- intersection and union closure can be obtained by taking a generalization of the product of type-automata,
- complement uses a generalized notion of determinism or finite models.

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Given two signature Σ_1 and Σ_2, $\mathcal{H}$ homomorphism from Σ_1 to Σ_2 if $\mathcal{H}$ is a pair of functions (g, h) such that:

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Theorem: Given Σ_1, Σ_2 two HOS and $\mathcal{H}$ a homomorphism from Σ_1 to Σ_2, if R is a recognizable set of Σ_2 then $\mathcal{H}^{-1}(R)$ is also recognizable.

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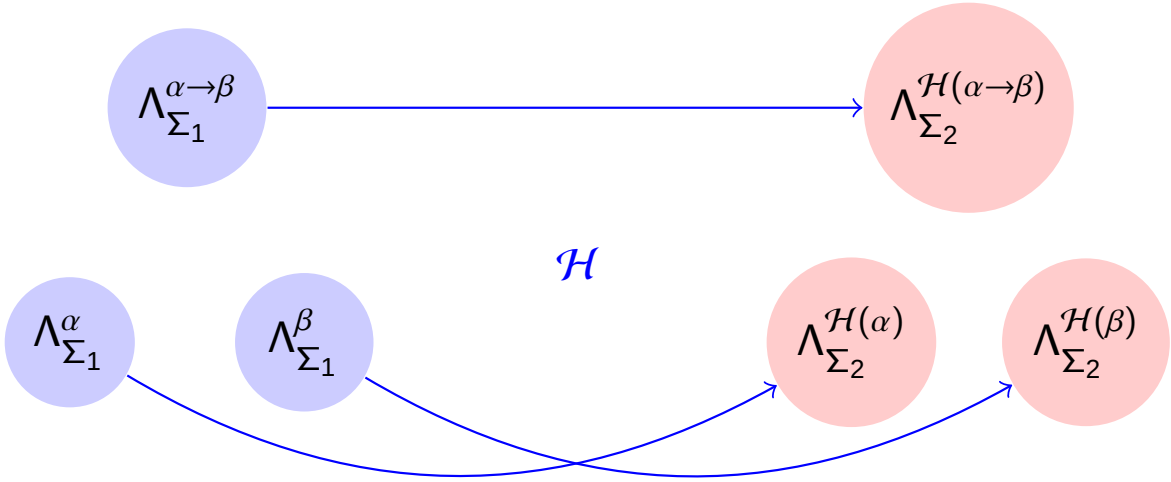

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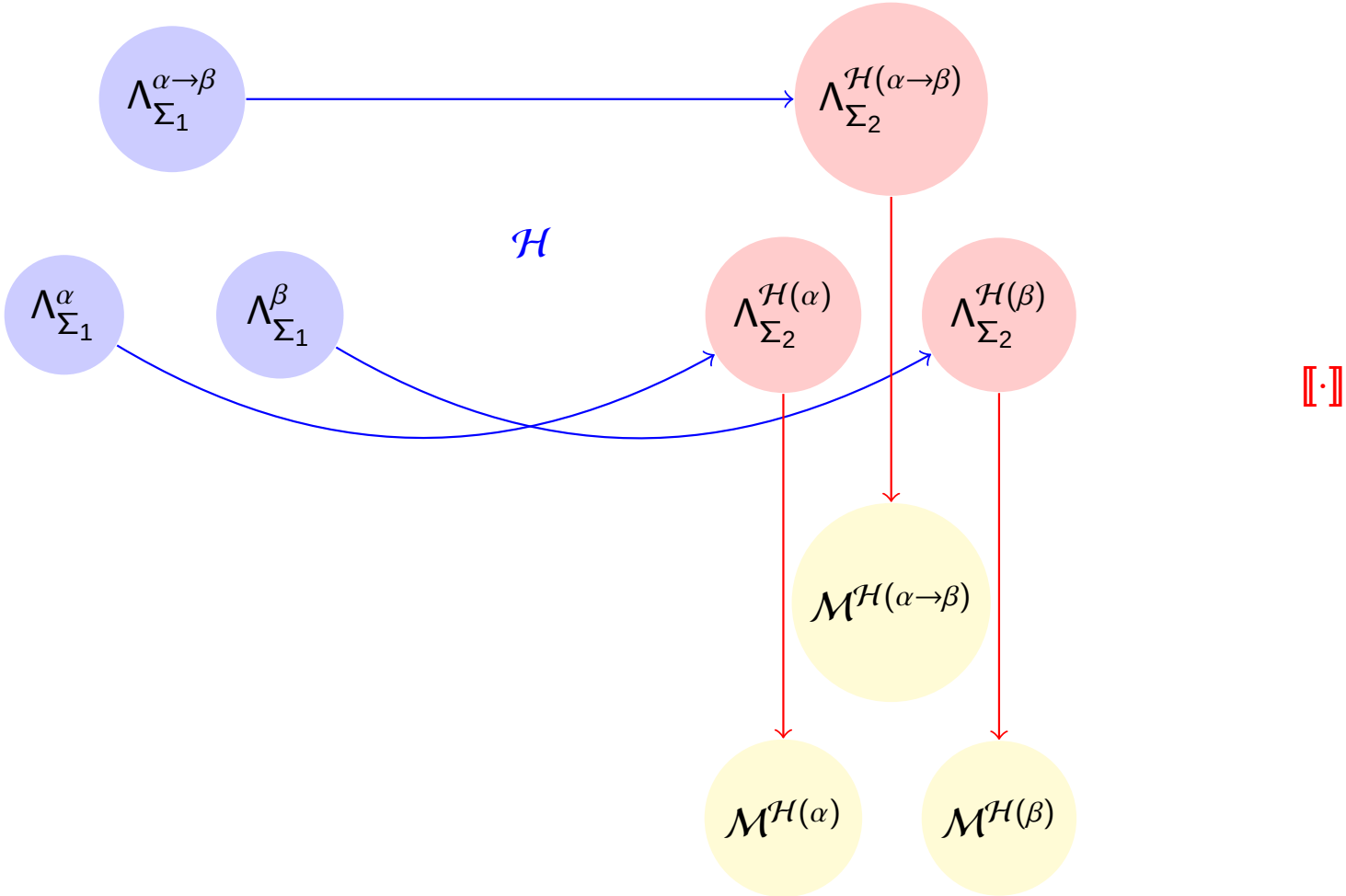

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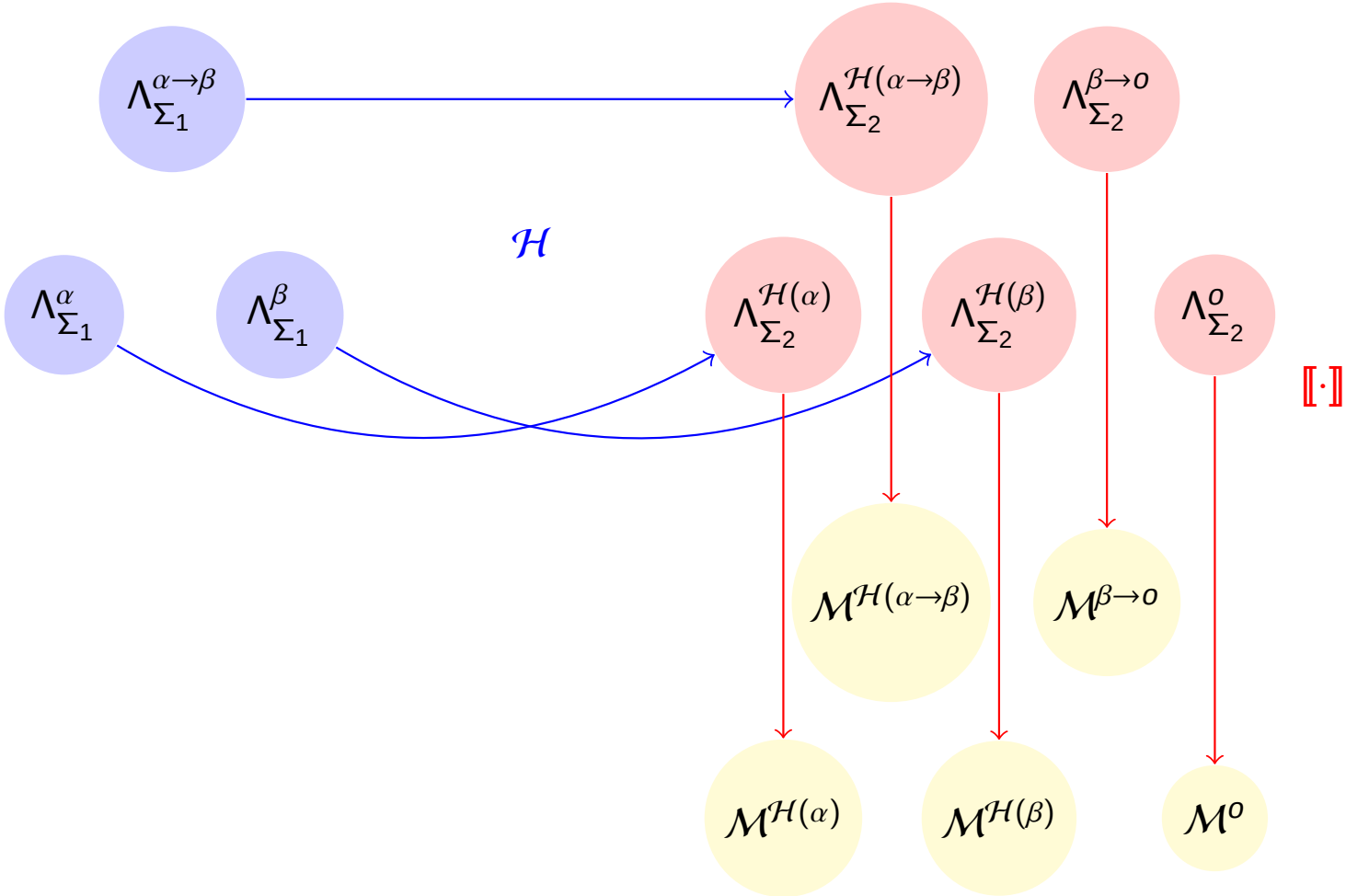

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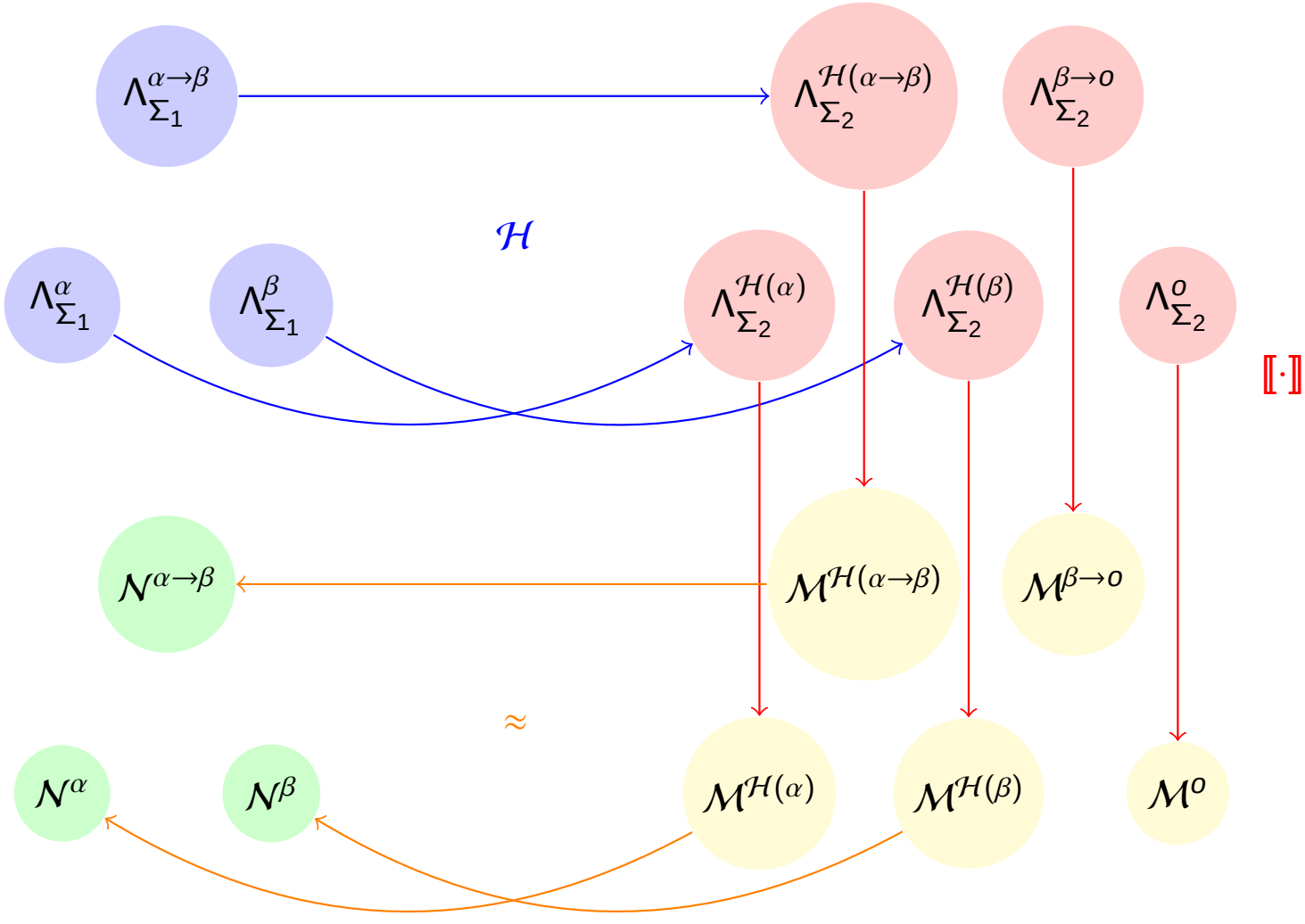

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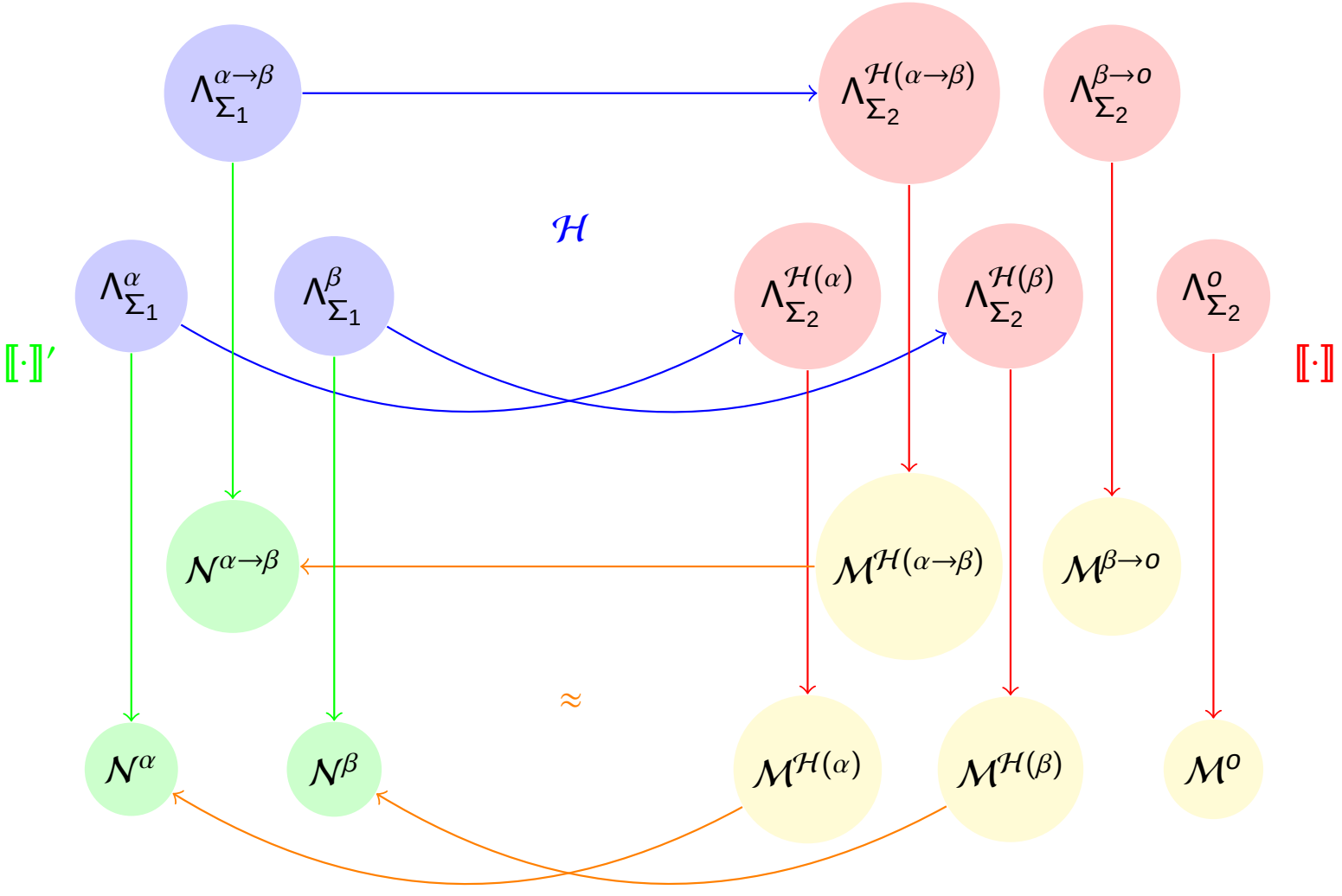

Outline

λ-context-free grammars

A context free grammar of lambda-term is a 4-tuple $(\Sigma_1, \Sigma_2, \mathcal{H}, S)$ where:

- Σ_1 is a second order signature, the *rule signature*

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Context free grammars of λ-terms subsume many formalisms like CFG, TAG, MCFG, PMCFG, MGs, IO-grammars,...

Parsing of λ-context-free grammars

Given a λ-CFG $\mathcal{G} = (\Sigma_1, \Sigma_2, \mathcal{H}, S)$:

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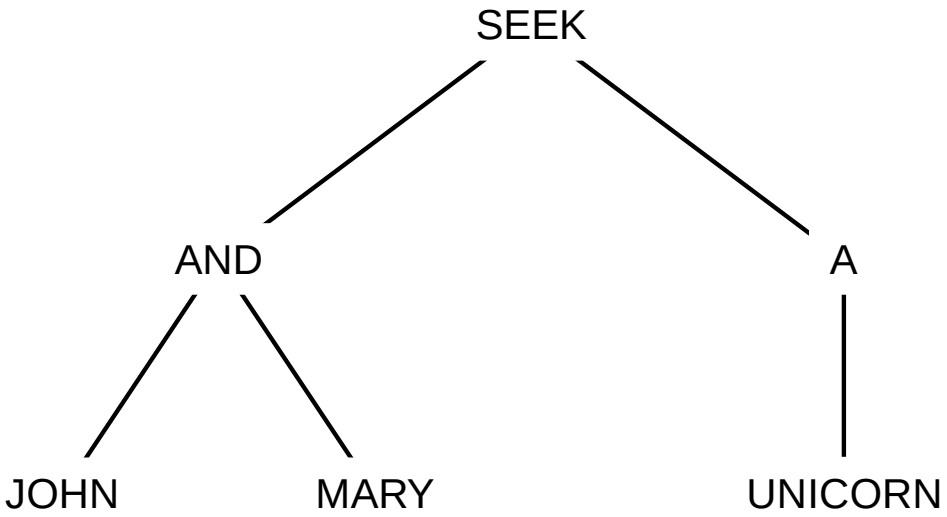
- this gives a simple generalization of Thatcher Theorem on the derivations of context free formalisms,
- furthermore this technique allows to show that parsing is a particular case of finding the inverse image of recognizable sets. The result extends to parsing recognizable sets.

Generation in Montague semantics



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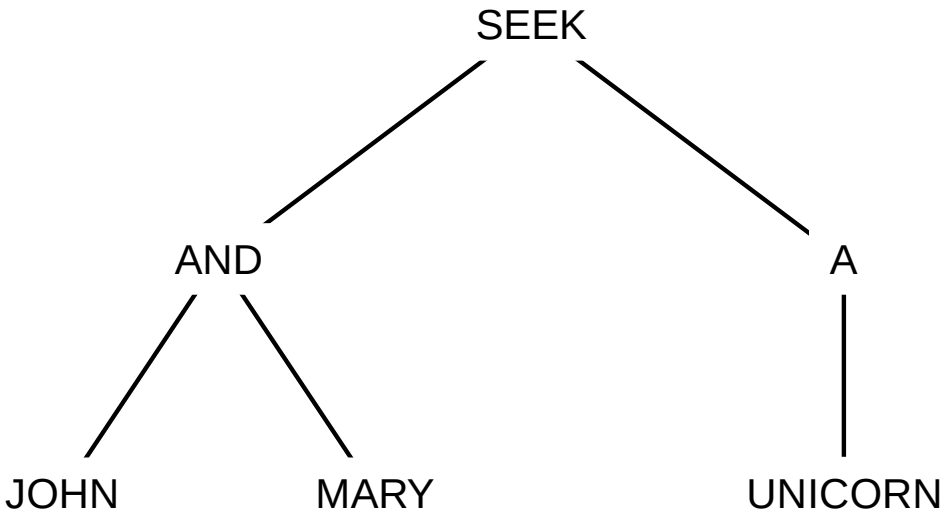

Jean et Marie cherchent une licorne

$$
\begin{aligned}
\text{SEEK} &= \lambda xy.x \text{ cherchent } y : np \to np \to s \\
\text{AND} &= \lambda xy.x \text{ et } y : np \to np \to np \\
\text{JOHN} &= \text{Jean} : np \\
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\text{A} &= \lambda x.\text{une } x : n \to np \\
\text{UNICORN} &= \text{licorne} : n
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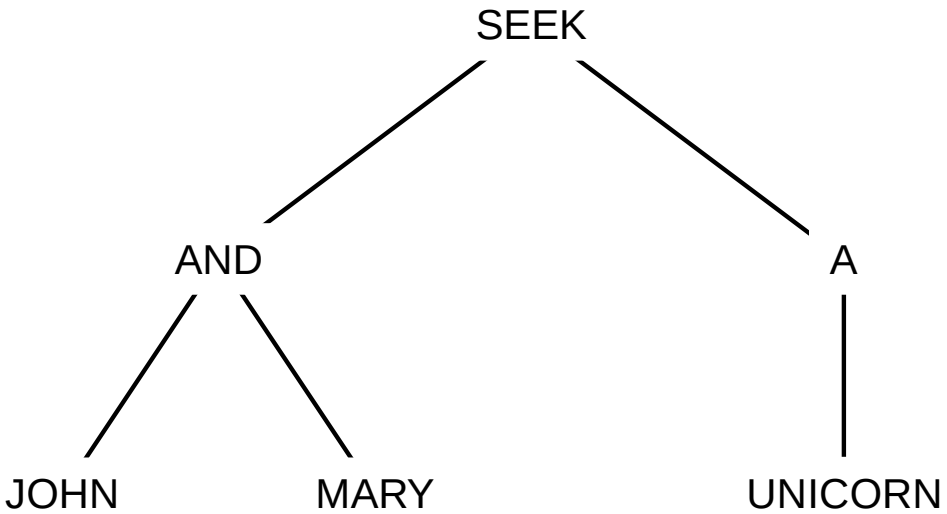

$$\exists(\lambda x.\ \wedge\ (\textsc{unicorn}\ x)(\wedge(\textsc{seek}\ \mathbf{j}\ x)(\textsc{seek}\ \mathbf{m}\ x)))$$
$$\approx$$
$$\exists x.\textsc{unicorn}(x) \wedge \textsc{seek}(\mathbf{j}, x) \wedge \textsc{seek}(\mathbf{m}, x)$$

$$\begin{aligned}
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\end{aligned}$$

Another corollary: decidability of 4th order matching

- From finite completeness we know that there is a model $\mathbb{M}$ and an element e of that model such that whenever $[\![t]\!]_{\mathbb{M}} = e$ then $t =_{\beta\eta} u$. Solving the equation in the model entails that the solution of a matching problem form a recognizable set.

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- **Lemma** The set of terms whose subterms only contain a bounded number of free variables can be described with a λ-context context free grammar.
- Decidability of fourth order matching becomes then a corollary of the closure of λ-context context free grammars under intersection with recognizable sets of λ-terms.

Outline

The other views on recognizability

Presently: λ-REC is defined with finite models (algebraic view) and intersection types (automaton view). The other views on recognizability are still to be explored.

- Congurential view: having canonical models that are only related to a language. The difficulty comes from the fact that standard models of the λ-calculus are not so well-behaved. Maybe we will need to turn to locally finite Cartesian Closed Categories.

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  - recognizable sets of λ-terms do not seem to be definable with a taylor-made MSOL (recognizable sets are not closed under relabeling)
  - a possibility is that modal μ-calculus is more adapted.
- Regular expressions: defining regular expressions of λ-terms is challenging since it requires to handle an unbounded number of free variables.

λ-calculus, higher-order programming schemes, and higher-order (collapsible) pushdown automata

Recognizable sets of λ-terms are well-behaved with context-free definitions and as such they have a neat connection with higher-order programming schemes.

- We already have a simple proof of Kobayashi's results for model-checking higher-order schemes.

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- An on-going work establishing the connection of Krivine machines with higher-order collapsible automata, may help along this line, it may also connect games on Krivine machines with games on higher-order (collapsible) automata.

Recognizability in the λ-calculus and profinite topology

In the literature on intersection types, Stone duality is a classical tool to construct models of the λ-calculus.

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- Another one consists in strudying Weil's pre-clones within the settled framework.
- Finally we may try to generalize the notion of implicit operations as defined by Almeida by using higher-order operations as definable by λ-terms.