

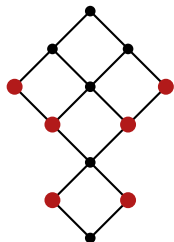
Calculs d'espaces duaux

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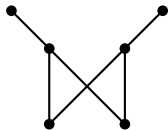
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Finite lattices and join-irreducibles

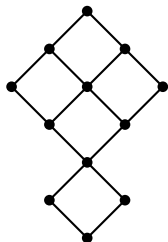
$x \neq 0$ is join-irreducible iff $x = y \vee z \Rightarrow (x = y \text{ or } x = z)$



D



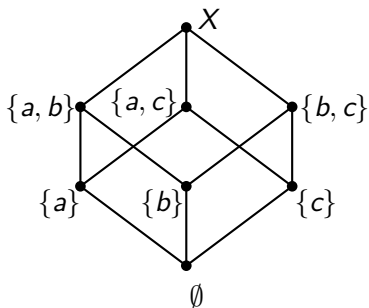
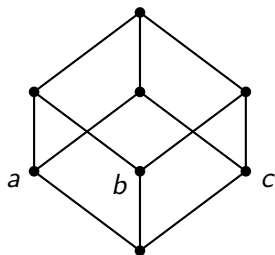
$J(D)$
Join-irreducibles



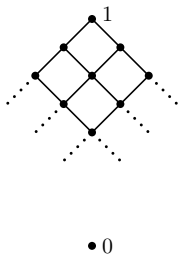
$\mathcal{D}(J(D))$
Downset lattice

The finite Boolean case

In the Boolean case, the **join-irreducibles** (J) are exactly the **atoms** (At) and the **downset lattice** (\mathcal{D}) of the atoms is just the **power set** (\mathcal{P})



Not enough join-irreducibles



$D = \mathbf{0} \oplus (\mathbb{N}^{op} \times \mathbb{N}^{op})$ has **no** join irreducible elements whatsoever!

Adding limits: prime filters

Prime filters are subsets witnessing missing join-irreducibles (i.e. maximal searches for join-irreducibles)

$F \subseteq D$ is a **filter** provided

- ▶ $1 \in F$ and $0 \notin F$
- ▶ $a \in F$ and $a \leq b \in D$ implies $b \in F$
- ▶ $a, b \in F$ implies $a \wedge b \in F$

(Order dual notion is that of an **ideal**)

A filter $F \subseteq D$ is **prime** provided, for $a, b \in D$

$$a \vee b \in F \implies (a \in F \text{ or } b \in F)$$

Points of the dual space

Let D be a DL and $F \subseteq D$. The following conditions are equivalent:

- ▶ F is a prime filter
- ▶ $D - F$ is a prime ideal
- ▶ $\chi_F : D \rightarrow 2$ is a lattice homomorphism

If, in addition, D is finite, then the above are equivalent to

- ▶ $F = \uparrow p$ where $p \in J(D)$

The Stone representation theorem

Let D be a distributive lattice, X_D the set of prime filters of D , then

$$\begin{aligned}\hat{}: D &\rightarrow \mathcal{P}(X_D) \\ a &\mapsto \hat{a} = \{F \in X_D \mid a \in F\}\end{aligned}$$

is an injective lattice homomorphism

- ▶ It is easy to verify that $\hat{}$ preserves $0, 1, \wedge$, and \vee
- ▶ Injectivity uses **Stone's Prime Filter Theorem**:

Let I be an ideal of D and F a filter of D . If $I \cap F = \emptyset$, then there is a prime filter F' with $F \subseteq F'$ and $I \cap F' = \emptyset$

Stone duality

Let D be a DL, the **Stone dual** of D is

$$(X_D, \sigma_D)$$

where $\sigma_D = \langle \hat{a} \mid a \in D \rangle$

then $D \cong \text{Im}(\hat{}) = \text{CompactOpen}(X)$ and

$$h : D \rightarrow E \quad \iff \quad h^{-1} : X_E \rightarrow X_D$$

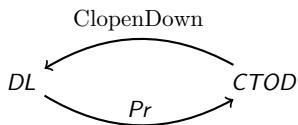
It is awkward to state what the dual category is ...

Priestley duality

Let D be a DL, the **Priestley dual** of D is

$$(X_D, \pi_D, \leq_\sigma)$$

where $\pi_D = \langle \hat{a}, (\hat{a})^c \mid a \in D \rangle$. The image of $\hat{}$ is characterized as the **clopen downsets**



The dual category are the spaces that are:

C = Compact TOD = **totally order disconnected**:

$$\forall x, y \quad (x \not\leq y \implies [\exists U \in \text{ClopDown}(X) \ y \in U \text{ but } x \notin U])$$

with continuous and order preserving maps

Categorical dualities

subalgebras

\longleftrightarrow

quotient spaces

quotient algebras

\longleftrightarrow

(closed) subspaces

products

\longleftrightarrow

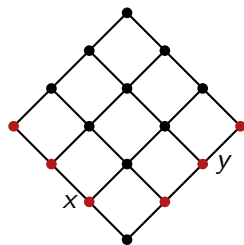
sums (disjoint union)

sums (coproduct)

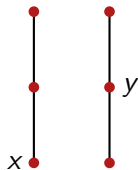
\longleftrightarrow

products

Products of lattices correspond to sums



$$D_1 \times D_2$$
$$D_1 = D_2 = 4$$

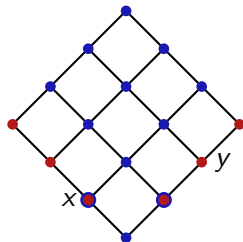


$$J(D) = J(D_1) \uplus J(D_2)$$
$$J(D_1) = J(D_2) = 3$$

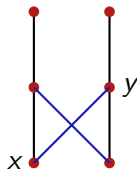
Sublattices correspond to order-quotients

For a sublattice D , the corresponding quasi-order is given by

$$x \preceq y \iff \forall d \in D (y \leq d \implies x \leq d)$$



$$D \leq D_1 \times D_2$$

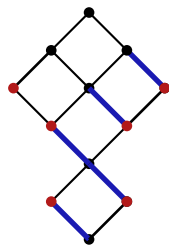


$$J(D) = (J(D_1) \uplus J(D_2)) / \preceq$$

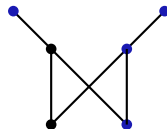
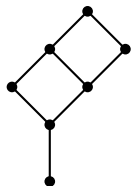
Lattice quotients correspond to subspaces

For a congruence θ , the corresponding subspace is given by

$$x \in X \iff \forall a, b \in D [a/\theta \leq b/\theta \implies (x \leq a \implies x \leq b)]$$

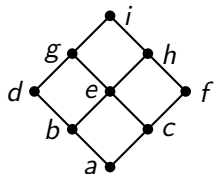


$D \longrightarrow D/\theta$

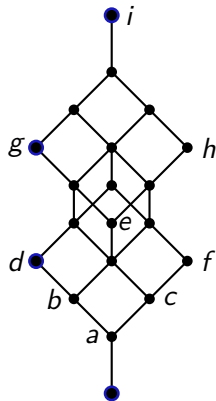


$X \subseteq J(D)$

Lattice coproducts correspond to products



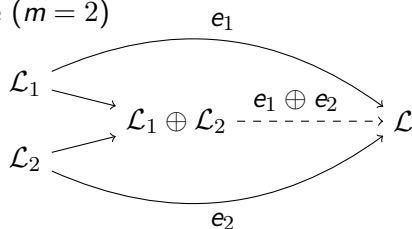
$$X \times Y$$
$$X = Y = 3$$



$$D \oplus E$$
$$D = E = 4$$

Calculating complex dual spaces I

- ▶ Find sublattices $\{\mathcal{L}_i\}_{i=1}^m$ whose union generates \mathcal{L} for which we can calculate the duals X_i
- ▶ Then we have ($m = 2$)

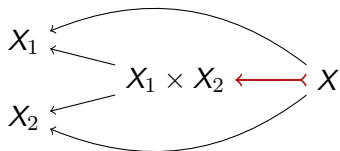


- ▶ The fact that the union of the $\{\mathcal{L}_i\}_{i=1}^m$ generates implies that

$$\mathcal{L}_1 \oplus \mathcal{L}_2 \xrightarrow{e_1 \oplus e_2} \mathcal{L}$$

Calculating complex dual spaces II

Dually we have

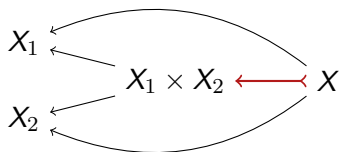


X is the (closed) subspace of $\prod_i X_i$ consisting of all those points $\bar{x} = (x_1, \dots, x_m)$ such that (for $m = 2$)

$$\forall M_1, M_2 \in \mathcal{L}_1 \oplus \mathcal{L}_2 \left[(e_1 \oplus e_2)(M_1) \leq (e_1 \oplus e_2)(M_2) \right. \\ \left. \implies \left((x_1, x_2) \in \hat{M}_1 \implies (x_1, x_2) \in \hat{M}_2 \right) \right].$$

Calculating complex dual spaces II

Dually we have



X is the (closed) subspace of $\prod_i X_i$ consisting of all those points $\bar{x} = (x_1, \dots, x_m)$ such that

$$\forall L_i, L_{ij} \in \mathcal{L}_i, j = 1, \dots, n \left[\left(\bigwedge_i L_i \leq \bigvee_j (\bigwedge_i L_{ij}) \text{ and } \bar{x} \in \bigwedge_i \hat{L}_i \right) \right. \\ \left. \implies \exists j \bar{x} \in \bigwedge_i \hat{L}_{ij} \right].$$

The dual of $\text{FO}[\mathcal{N}, \mathbf{a}]/\equiv_{A^*}$

NB! **No order** in the language of the logic

This corresponds to the BA generated by the languages, for $a \in A$ and $P \subseteq \mathbb{N}$,

$$L_P^a = \{w \in A^* \mid c_a(w) \subseteq P\} \text{ and } L_P = \{w \in A^* \mid |w| \in P\},$$

where $c_a(w) = \{k \in \mathbb{N} \mid w_k = a\}$

We calculate the Priestley dual of $\mathcal{L} = \langle L_P^a, L_P \mid a \in A \text{ and } P \subseteq \mathbb{N} \rangle$.
The dual of the Boolean algebra generated by these sets is the underlying space of this ordered space!

Strategy for calculating the dual of \mathcal{L}

- ▶ We calculate the dual X_a of $\mathcal{L}_a = \langle L_P^a \mid P \subseteq \mathbb{N} \rangle$
- ▶ We calculate the dual X_B of $\mathcal{L}_B = \langle \bigcup \{ \mathcal{L}_a \mid a \in B \} \rangle$ for $B \subsetneq A$
- ▶ We calculate the dual X_A of \mathcal{L}_A
- ▶ We calculate the dual Y of $\mathcal{K} = \langle L_P \mid P \subseteq \mathbb{N} \rangle$
- ▶ We calculate the dual X of $\mathcal{L} = \langle \mathcal{L}_A \cup \mathcal{K} \rangle$

The dual of $\mathcal{L}_a = \langle L_P^a \mid P \subseteq \mathbb{N} \rangle$

A direct computation (assuming $|A| \geq 2$):

- ▶ For $P, Q \in \mathcal{P}(\mathbb{N})$ we have

$$L_P^a \subseteq L_Q^a \iff P \subseteq Q \quad \text{and} \quad L_P^a \cap L_Q^a = L_{P \cap Q}^a.$$

\therefore Every language in \mathcal{L}_a is a finite union of L_P^a s

- ▶ For each $P \in \mathcal{P}(\mathbb{N})$ the language L_P^a is join-prime in \mathcal{L}_a
- ▶ There is a one-to-one correspondence between the set of prime filters of \mathcal{L}_a and the set of filters of $\mathcal{P}(\mathbb{N})$

The dual of \mathcal{L}_a

The underlying set of the dual of \mathcal{L}_a is

$$X_a = \text{Filt}(\mathcal{P}(\mathbb{N}))$$

The topology is given by

$$\tau = \langle \hat{P} \mid P \in \mathcal{P}(\mathbb{N}) \rangle$$

where

$$\hat{P} = \{F \in \text{Filt}(\mathcal{P}(\mathbb{N})) \mid P \in F\}$$

The dual space of \mathcal{L}_a is the Vietoris space of the Stone-Čech compactification, $\beta(\mathbb{N})$, of the discrete space on \mathbb{N}

The dual of \mathcal{L}_B for $B \subsetneq A$

Let $B = \{b_1, \dots, b_n\}$, and let $a \in A - B$

▶ $\mathcal{L}_B = \langle \bigcup \{\mathcal{L}_b \mid b \in B\} \rangle$ so X_B is a subspace of X^m where $X = \text{Filt}(\mathcal{P}(\mathbb{N}))$

▶ $\bar{F} = (F_1, \dots, F_m) \in X_B$ iff for all $M_1, M_2 \in \bigoplus_i \mathcal{L}_{b_i}$

$$\left[\bar{F} \in M_1 \text{ and } (\bigoplus_i e_i)(M_1) \subseteq (\bigoplus_i e_i)(M_2) \right] \implies \bar{F} \in M_2$$

The M s are finite unions of $\prod_i \widehat{L_{P_i}^{b_i}}$ s and $(\bigoplus_i e_i)$ of such a set is $L(\bar{P}) := \bigcap_i L_{P_i}^{b_i}$

WLOG $(\bigoplus_i e_i)(M_1) = L(\bar{P})$ but $(\bigoplus_i e_i)(M_2) = \bigcup_j L(\bar{P}_j)$

The dual of \mathcal{L}_B for $B \subsetneq A$

So we get $\bar{F} \in X_B$ iff for all $\bar{P}, \bar{P}_1, \dots, \bar{P}_n \in (\mathcal{P}(\mathbb{N}))^m$

$$\left[\bar{P} \in \bar{F} \text{ and } L(\bar{P}) \subseteq \bigcup_j L(\bar{P}_j) \right] \implies \exists j \bar{P}_j \in \bar{F}$$

For $F \in \text{Filt}(\mathcal{P}(\mathbb{N}))$, let the **content** of F be

$$C(F) = \bigcap F$$

Theorem

$$X_B = \{ \bar{F} \mid \{C(F_i)\}_{i=1}^m \text{ are pairwise disjoint} \}$$

Join-irreducibility of the $L(\bar{P})$ s

A **decomposition** of a set S is a collection of pairwise disjoint subsets of S whose union is S . Let $\bar{P} \in \mathcal{P}(\mathbb{N})^m$ and let $S \subseteq \bigcup_{i=1}^m P_i$, then

$$L(\bar{P}) = \bigcup \{L((P_i - S) \cup S_i)_{i=1}^m \mid S_i \subseteq P_i \text{ and } \{S_i\}_{i=1}^m \text{ is an } m\text{-decomposition of } S\}.$$

Lemma

$L(\bar{P})$ is join irreducible iff $\{P_i\}_{i=1}^m$ are pairwise disjoint

Some lemmas

Let $\bar{P} \in \mathcal{P}(\mathbb{N})^m$ and $\{\bar{P}_j\}_{j=1}^n \subseteq \mathcal{P}(\mathbb{N})^m$ and suppose that

$$L(\bar{P}) \subseteq \bigcup_{j=1}^n L(\bar{P}_j)$$

Lemma $L(\bar{P}) \subseteq \bigcup \{L(\bar{P}_j) \mid P_i - P_{ij} \text{ are finite for each } i\}$

Lemma *There is a finite set $S \subseteq \mathbb{N}$ such that, for each decomposition $\{S_i\}_{i=1}^m$ of S with $S_i \subseteq P_i$, there is $j \in \{1, \dots, n\}$ with*

$$L(\bigcup_{i=1}^m ((P_i - S) \cup S_i)) \subseteq L(\bar{P}_j).$$

Proof.

Take $S = \bigcup \{P_i - P_{ij} \mid 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$



Proof sketch

$\bar{F} \in X_B \iff \{C(F_i)\}_{i=1}^m$ are pairwise disjoint

- ▶ If $n \in C(F_k) \cap C(F_l)$, with $k \neq l$, take M_1 with $(\oplus e_i)(M_1) = L_{P_k}^{b_k} \cap L_{P_l}^{b_l}$ where $P_k \in F_k$ and $P_l \in F_l$, and use M_2 so that

$$(\oplus e_i)(M_2) = (L_{P_k - \{n\}}^{b_k} \cap L_{P_l}^{b_l}) \cup (L_{P_k}^{b_k} \cap L_{P_l - \{n\}}^{b_l})$$

- ▶ For converse, get S finite from lemma 2, and use $\{S_i\}$ with $S \cap C(F_i) \subseteq S_i \subseteq P_i$ and remove each $s \in S - S_i$ from P_i to get a $P'_i \in F_i$

The dual space of \mathcal{L}_A

Theorem

$\bar{F} \in X_A$ iff *one* of the following conditions hold:

- ▶ Each $F_i = \uparrow P_i$ and $\{P_i\}_{i=1}^m$ is a decomposition of $\downarrow n$ for some $n \in \mathbb{N}$
- ▶ $\{C(F_i)\}_{i=1}^m$ is a decomposition of \mathbb{N}

The points of the first kind are in one-to-one correspondence with A^* via

$$w \mapsto \bar{F}_w = (\uparrow c_a(w))_{a \in A}$$

The dual space of \mathcal{K}

Recall $\mathcal{K} = \langle L_P \mid P \subseteq \mathbb{N} \rangle$

where $L_P = \{w \in A^* \mid |w| \in P\}$

Theorem

$\mathcal{K} \cong \mathcal{P}(\mathbb{N})$ so the dual of \mathcal{K} is $Y \cong \beta(\mathbb{N})$

The dual space of \mathcal{L}

Theorem

$(\bar{F}, \mu) \in X$ iff **one** of the following conditions hold:

- ▶ Each $F_i = \uparrow P_i$ and $\{P_i\}_{i=1}^m$ is a decomposition of $\downarrow n$ for some $n \in \mathbb{N}$ and $\mu = \uparrow\{n\}$
- ▶ $\{C(F_i)\}_{i=1}^m$ is a decomposition of \mathbb{N} and μ is a non-principal ('free') ultrafilter of $\mathcal{P}(\mathbb{N})$

So

$$X = A^* \cup ((X_A - A^*) \times (\beta(\mathbb{N}) - \mathbb{N}))$$

Further work

- ▶ understand the generalised monoid operation on X and what properties it has generalising the properties of $\mathcal{L} \cap \text{Reg}$
- ▶ add \leq to the logic and at least get dual of two quantifier fragment (i.e. apply block product to something like what we have done here)
- ▶ compute profinite monoids known to lie in a product, by the quotient-of-a-coproduct versus sub-of-a-product duality