On random subgroups of free groups

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Outline

- Preliminaries
- Random subgroups given by a tuple of generators
- Random subgroups given by a Stallings graph
- Generalizations of the tuple model: Markovian automata
- Generalizations of the tuple model: Bernoulli-like models

Random generation of subgroups of free groups 1/2

- **A variety of motivations**

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- The generic properties of subgroups of certain other groups (towards cryptography)
- The generic properties of group presentations
- The generic properties of (finitely presented) groups (harder. . .)
- There are by now classic results in both directions: on subgroups and on presentations

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- One can pick a set of generators for the subgroup, or a set of relators for the presentation: Gromov (*most groups are hyperbolic*), Champetier, Ol'shanskiĭ, Arzhantseva, Ollivier, Miasnikov, Schupp, Shpilrain, Jitsukawa, and many more...

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- Or one can pick directly a Stallings graph: Bassino, Martino, Ventura, Nicaud, Weil

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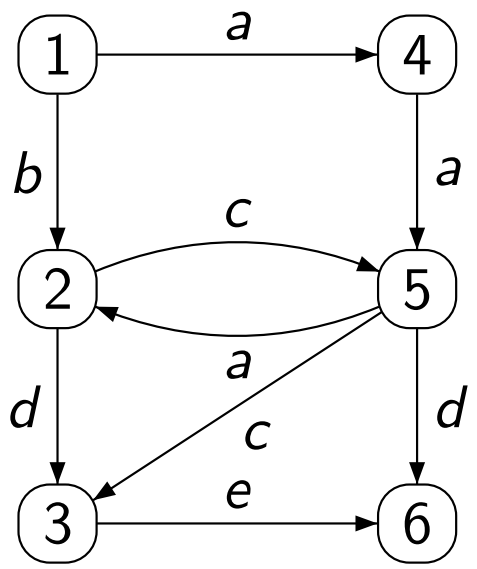

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- Property $C'(\frac{1}{6})$ implies hyperbolicity of $\langle A \mid \vec{h} \rangle$

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- Why not for presentations?

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- H is malnormal
- Restricting to cyclically reduced words: $C'(\lambda)$ for any $\lambda < 1$: $G = \langle A \mid \vec{h} \rangle$ is hyperbolic

The density model (Gromov, Champetier, Olshanskii, Ollivier, . . .): exponential number of relators

- Fix $d < 1$, draw uniformly at random a tuple $\vec{h} = (h_1, \ldots, h_k)$ of density d in the sphere of radius n: roughly $(2r-1)^{dn}$ cyclically reduced words of length n, consider $G = \langle A \mid \vec{h} \rangle$ as n tends to ∞

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- In fact, if $d < \frac{1}{2}$, G is hyperbolic — by a higher-dimensional counting argument (on van Kampen diagrams)

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- But... how *do* we draw a Stallings graph at random?

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- Draw a size m of an orbit, decide whether it is a cycle or a sequence; and draw another random partial injection of size $n - m$

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- Either situation is generically negligible, so a rejection algorithm will work in an average of $1 + o(1)$ rounds [BNW]

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- Expected number of a-labeled edges: $n - \sqrt{n}$ (whereas in the word-based distribution, Stallings graph have few edges)
- Expected rank of H:

$$\mathbb{E}(rank(H)) = \mathbb{E}(edges) - n + 1 = (r-1)n - r\sqrt{n} + 1$$

with standard deviation $o(\sqrt{n})$

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- Malnormality is negligible, with probability $\mathcal{O}(n^{-\frac{r}{2}})$

Groups presented by a Stallings graph are generically trivial

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- and add probability weights on the transitions – in such a way that, at each state q, the sum of probabilities of the transitions out of state q is equal to 1.
- For each n, this gives a probability law on the set of words of length n

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- Example: the (uniform) distribution on the geodesics of $PSL_2(\mathbb{Z}) = \langle a, b \mid a^2, b^3 \rangle$: all reduced words without a squared letter
- Usually require ergodicity of the Markovian automaton, that is, *strong connectedness* and *the loops have relatively prime lengths*

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- $\Gamma(H)$ has a central tree of linear height, and consists of one loop for each h_i, connecting the leaves of the tree

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- All this *exponentially generically* if $\vec{h}$ consists of polynomially many reduced words

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- super-polynomially generically if $\log n = o(\mu(n)$ (*e.g.* $\log^{1+\varepsilon} n$, or $\log n \log \log n$, or ...)

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- Examples: all reduced words; all geodesics of $PSL(2, \mathbb{Z})$
- Two parameters seem to play a role: $\vec{p}_{max} = \max_{b \in B} p_b$ and $\vec{p}_e = \prod_{b \in B} p_b^{p_b}$ (which is $-\log(entropy)$ of the Bernoulli source). Note that $\vec{p}_e \leq \vec{p}_{max}$ and the two are equal if $\vec{p}$ is uniform

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- And between $\vec{p_e}^{\,-\frac{n}{12}}$ and $\vec{p_e}^{\,-\frac{n}{2}}$?

Thank you for your attention!