The separation problem for languages of finite words

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Based on Lorijn's slides

Het probleem van de scheiding van talen

- Motivaties voor deze vraag
  - Een eenvoudige klasse van afscheiders maakt algoritmische efficintie
    Voorbeeld: toegankelijkheid teller systemen.
  - hergroepering het lidmaatschap probleem.

Outline

Separation problem

- Let Sep be a class of languages closed under complement.
- $L_1, L_2 \in \mathcal{C}$ are Sep-**separable** if $\exists K \in \mathrm{Sep}$ st $L_1 \subseteq K$ and $L_2 \cap K = \varnothing$.

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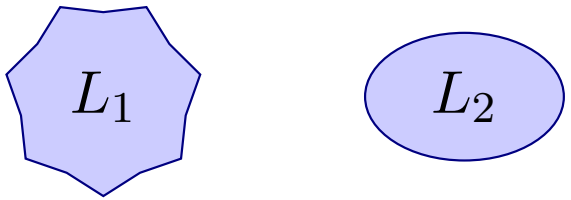

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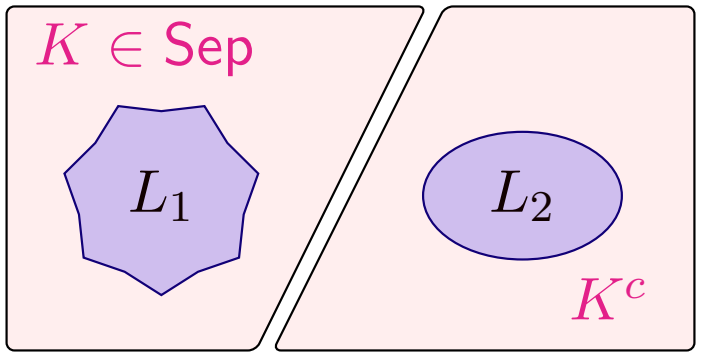

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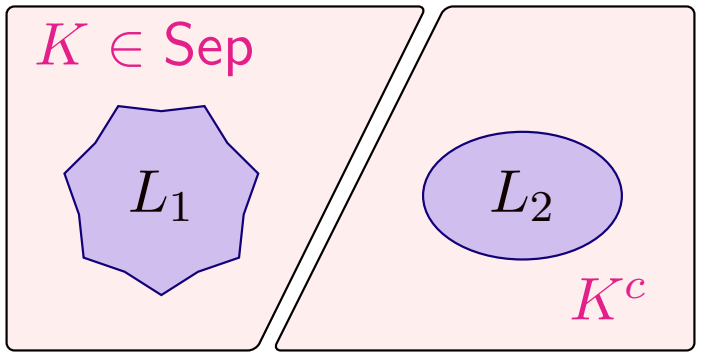

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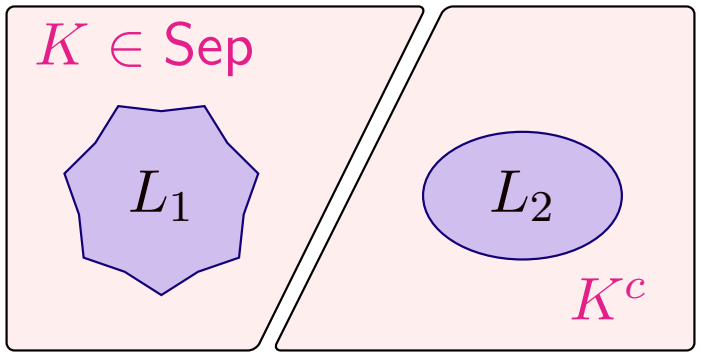

- **No minimal** separator in general.
- **Co-example**: $(a^2)^*$ and $a(a^2)^*$ are disjoint but not $\mathrm{FO}(<)$-separable.

Separation problem: 2 questions

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   "Given $L_1, L_2 \in \mathcal{C}$, decide whether they are Sep-separable"

   - If $\mathcal{C}$ complement-closed and $L \in \mathcal{C}$:

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2. Compute a separator.

3. How costly is it to decide? to compute a separator?

Why separation?

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  - in math (eg, Hahn-Banach, optimization),
  - in CS: approximate query answering, [Czerwiński–Martens–Masopust13]
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- Algorithmic effectiveness, eg, interpolation techniques in verification.
- It's fun!

Separation problem vs. profinite approach

- Assume $\mathcal{C} = \mathrm{Reg}$ and Sep is a variety of languages.

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- Theorem [Almeida96] L_1, L_2 are Sep-separable $\iff$ $\overline{L}_1 \cap \overline{L}_2 = \varnothing$.
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- Deciding $\overline{L}_1 \cap \overline{L}_2 = \varnothing$ is equivalent to computing the 2-pointlikes for Sep

Separation problem vs. profinite approach

- Computing the 2-pointlikes is not easy in general.
- Known to be computable for many classes
  - groups [Ash91, RZ92, Auinger04, Auinger–Steinberg05]
  - aperiodic monoids [Henckell88, Henckell–Rhodes–Steinberg10]
  - J- and R-trivial monoids [Almeida–Z.95,Almeida–Costa–Z.08]
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- Papers read/understood mainly by their referees/authors.

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Piecewise testable languages

- Want to separate regular by piecewise testable languages.
- u is a subword of v if

$$u = a_1 \cdots a_n \qquad v = v_0 a_1 v_1 \cdots v_{n-1} a_n v_n, \qquad a_i \in A, \qquad v_i \in A^*$$

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- u, v are $\sim_n$-**equivalent** if they have the same subwords up to length n.
- A piecewise-testable language is a union of $\sim_n$-classes.

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Co-example. $(ab)^*$ is not PT.

Separability: co-examples

- $(ab)^+$ and $(ba)^+$ are not PT-separable.

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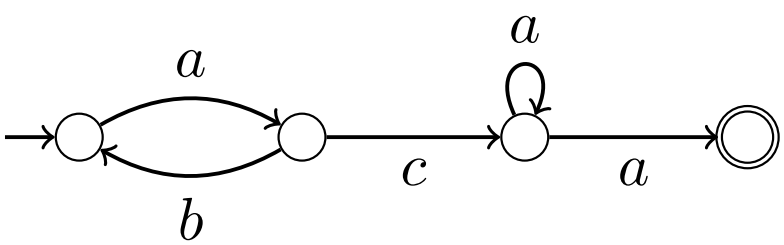
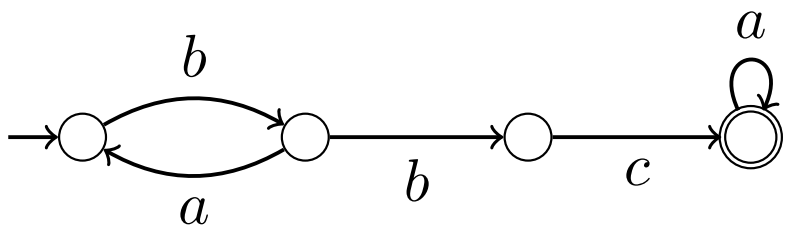
- The languages recognized by the following DFAs are not PT-separable.





Main result for PT-separability

Theorem Given NFAs $\mathcal{A}_1, \mathcal{A}_2$, one can determine in PTIME$(|Q_1|, |Q_2|, |A|)$, whether $L(\mathcal{A}_1)$ and $L(\mathcal{A}_2)$ are PT-separable.

Notes

- Obvious semi-algorithm for testing separability.
- Need a witness for non-separability.
- 2 independent proofs
  - Czerwiński–Martens–Masopust.
  - Place-van Rooijen–Z.

$(\vec{u}, \vec{B})$-paths

A **factorization pattern** is a pair $(\vec{u}, \vec{B})$, where

$$\vec{u} = (u_0, \ldots, u_p) \quad \text{with} \quad u_0, \ldots, u_p \in A^*,$$
$$\vec{B} = (B_1, \ldots, B_p) \quad \text{with} \quad B_1, \ldots, B_p \text{ nonempty subalphabets of } A.$$

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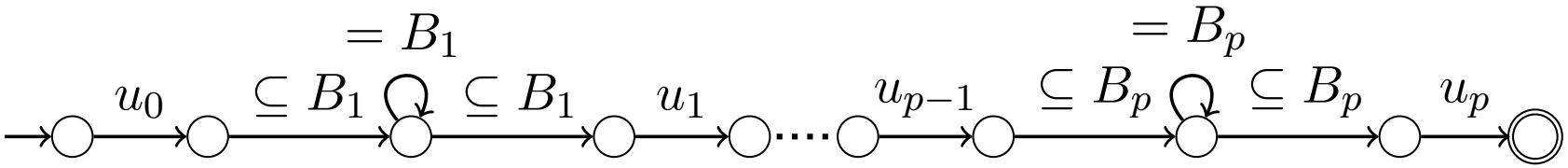
A $(\vec{u}, \vec{B})$-**path** in $\mathcal{A}$ is a successful path, of the form:



Proof of the theorem

Proposition 1 $L(\mathcal{A}_1)$ and $L(\mathcal{A}_2)$ are **not PT-separable**
$\exists\, (\vec{u}, \vec{B})$ such that both $\mathcal{A}_1$ and $\mathcal{A}_2$ have a $(\vec{u}, \vec{B})$-path.

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- By adding meta-transitions in $\mathcal{A}_i$, finding a $(\vec{u}, \vec{B})$-witness reduces to:

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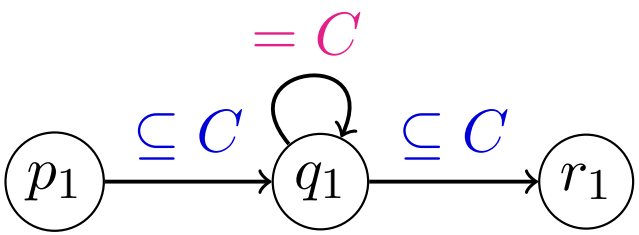

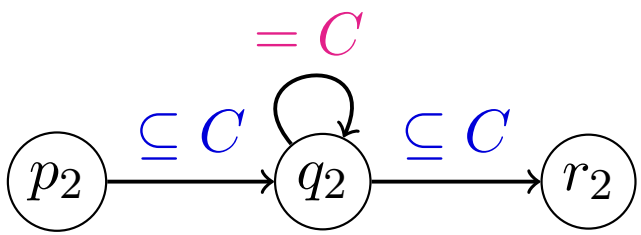

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- This is in PTIME: iteratively use Tarjan's algorithm.

Detecting forbidden patterns

- If C exists, $C \subseteq C_1 \stackrel{\text{def}}{=} \mathsf{alph\_scc}(q_1, \mathcal{A}_1) \cap \mathsf{alph\_scc}(q_2, \mathcal{A}_2)$ (LINEAR)
- Restrict the automata to alphabet C_1, and repeat the process:

$$C_{i+1} \stackrel{\text{def}}{=} \mathsf{alph\_scc}(q_1, \mathcal{A}_1 \restriction_{C_i}) \cap \mathsf{alph\_scc}(q_2, \mathcal{A}_2 \restriction_{C_i}).$$

- After $n \leqslant |A|$ iterations, $C_n = C_{n+1}$.
  - If $C_n = \varnothing$, the answer is no.
  - If $C_n \neq \varnothing$, it is the maximal possible C with $(= C)$-loops around q_1, q_2.
- Then, determine the remaining paths. (LINEAR)
- Overall LINEAR algorithm.

Separability reduces to finding $(\vec{u}, \vec{B})$-paths

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- Easy. If there are such $(\vec{u}, \vec{B})$-paths, then

$$\forall n,\ \exists u_n \in L(\mathcal{A}_1),\quad w_n \in L(\mathcal{A}_2) \quad \text{s.t.} \quad u_n \sim_n w_n.$$

Hence, $L(\mathcal{A}_1)$ and $L(\mathcal{A}_2)$ are **not separable**.

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- $C^{\circledast} =$ set of words with alphabet exactly C.

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- $(w_n)_n$ is $(\vec{u}, \vec{B})$-adequate if

$$\forall n \geq 0,\ w_n \in \mathcal{L}(\vec{u}, \vec{B}, n).$$

- $(w_n)_n$ is adequate if it is $(\vec{u}, \vec{B})$-adequate for some $(\vec{u}, \vec{B})$.

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- Lemma 1 Every sequence $(w_n)n$ of words has an adequate subsequence.
  Immediate from Simon's Factorization Forest Theorem.

- Lemma 2 If $(v_n)_n$ and $(w_n)_n$ are such that
  - $(v_n)_n$ is $(\vec{u}, \vec{B})$-adequate
  - $(w_n)_n$ is $(\vec{t}, \vec{C})$-adequate
  - $v_n \sim_n w_n$ for every $n \geq 0$.
  - (+ simple technical condition)

  Then, $\vec{u} = \vec{t}$ and $\vec{B} = \vec{C}$.

Computing a separator

Theorem Let $L_1 = L(\mathcal{A}_1)$ and $L_2 = L(\mathcal{A}_2)$. Let $p = \max(|Q_1|, |Q_2|) + 1$ and $\kappa = p|A|2^{2^{|A|}|A|(p|A|+1)}$. TFAE:

- L_1 and L_2 are PT-separable.
- L_1 and L_2 are $PT[\kappa]$-separable.
- The language $[L_1]_{\sim\kappa}$ separates L_1 from L_2.
- $(\mathcal{A}_1, \mathcal{A}_2)$ have no witness of non PT-separability.

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Ramsey+Pumping to show: if $\exists\, u_1 \in L_1$, $u_2 \in L_2$ and $u_1 \sim_\kappa u_2$, then

$$\forall n \quad \exists\, u_{1,n} \in L_1 \quad \exists\, u_{2,n} \in L_2 \qquad u_{1,n} \sim_n u_{2,n}.$$

Separation by $\mathrm{FO}^2(<)$-definable languages

Same power: $\mathrm{FO}^2(<)$, Unambiguous Languages (UL), Δ_2, $\Sigma_2 \cap \Pi_2$, DA, UTL [Schützenberger,Schwentick–Therien–Vollmer,Pin–Weil, Therien–Wilke].

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Theorem Let M_1 and M_2 be monoids recognizing $L_1, L_2 \subseteq A^*$.
Let $\kappa = (2|M_1||M_2| + 1)(|A| + 1)^2$. TFAE:

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Note

- New result (pointlike sets not known to be computable for DA).
- No algorithm in terms of forbidden patterns.

Separation by LT languages

Locally testable language: finite boolean combination of uA^*, A^*uA^*, A^*u.

$u \equiv_k v$ if u and v have the same factors of length $\leqslant k$ (+ prefix, suffix).

Theorem Let M_1 and M_2 be monoids recognizing $L_1, L_2 \subseteq A^*$.
Let $\kappa = 4(|M_1||M_2| + 1)$. TFAE:

- L_1 and L_2 are LT-separable.
- L_1 and L_2 are LT$[\kappa]$-separable.
- The language $[L_1]_{\equiv_\kappa}$ separates L_1 from L_2.

Separation by LT languages

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Theorem Forbidden pattern characterization.

- L_1, L_2 separable iff no two sublanguages of the form

$$u_1 v_1^* u_2 v_2^* \cdots v_{n-1}^* u_n$$

inducing same infinite prefix, infinite suffix, bi-infinite infixes $u_{i-1}^{-\infty} v_i u_{i-1}^{+\infty}$.

- Yields NP-hardness.

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- Efficient computation of separators.
- Algebraic object from L_1, L_2 on which one could test separability.