

Leaktight and simple probabilistic automata

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Joint work with: Hugo Gimbert

May 23, 2013

Probabilistic automata

- Generalization of DFA by Rabin 1963
- Transition matrices are stochastic
- Formally: $\mathcal{A} = (Q, A, i, (M_a)_{a \in A}, F)$
- Word $w = a_1 \dots a_n$ accepted with probability p if

$$\mathcal{A}(w) = \mathbf{i}M_w\mathbf{F} = p$$

where $M_w = M_{a_1} \dots M_{a_n}$

- Stochastic language with respect to a *cutpoint* $\lambda \in [0, 1]$ (ex: all words w s.t. $\mathcal{A}(w) > \lambda$)

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- Stochastic language with respect to a *cutpoint* $\lambda \in [0, 1]$ (ex: all words w s.t. $\mathcal{A}(w) > \lambda$)
- Similarly a PA on infinite words

Value 1 problem

Definition

A cutpoint $\lambda \in [0, 1]$ is isolated if there is $\epsilon > 0$ such that $\forall w \in A^*$ we have

$$|\mathcal{A}(w) - \lambda| > \epsilon$$

Problem (Isolated cutpoint problem)

Given \mathcal{A} and $\lambda \in [0, 1]$, decide whether λ is isolated.

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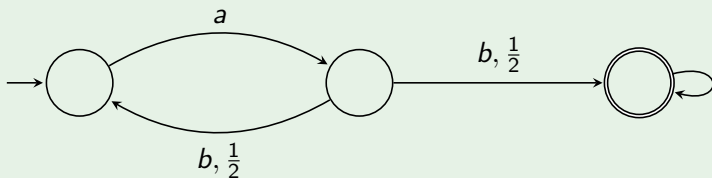
Problem (Isolated cutpoint problem)

Given \mathcal{A} and $\lambda \in [0, 1]$, decide whether λ is isolated.

- Isolated cutpoint problem is undecidable [BMT77,GO10]
- Isolated cutpoint problem for the special case of $\lambda = 1$ is called *value 1 problem*

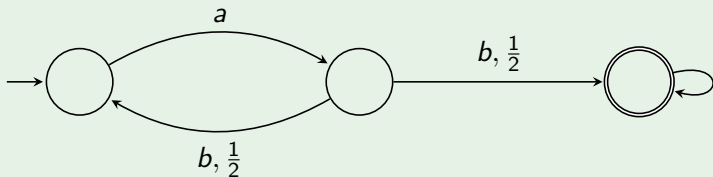
Example of an automaton with value 1

Example



Example of an automaton with value 1

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Has value 1 with the sequence of words $((ab)^n)_{n \in \mathbb{N}}$.

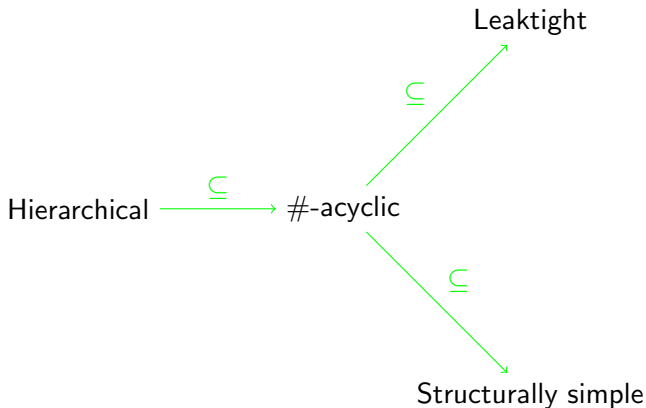
A few classes of PAs with decidable value 1 problem

- Goal: Find a robust class of PAs with decidable value 1 problem

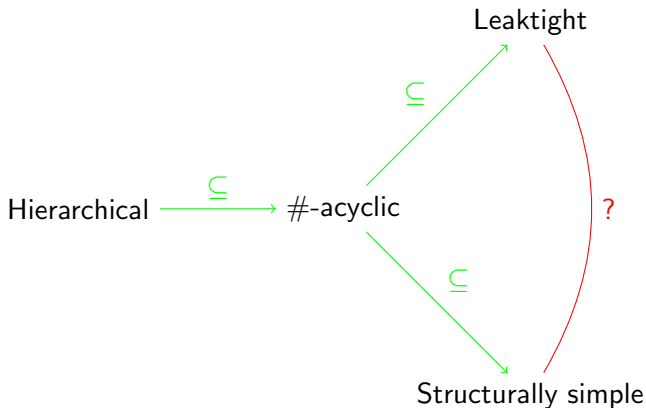
A few classes of PAs with decidable value 1 problem

- Goal: Find a robust class of PAs with decidable value 1 problem
- Hierarchical automata [CSV09]
- #-acyclic automata [GO10]
- Leaktight automata [FGO11]
- Simple automata [CT11]

A few classes of PAs with decidable value 1 problem



A few classes of PAs with decidable value 1 problem



Our question

Our question is: How are leaktight and structurally simple classes related?

Introduction (Simple automata)

- By Chatterje and Tracol, studied in the context of PAs on infinite words
- Relies on *Decomposition separation theorem* to show that:
- Limit problem for reachability and parity condition on structurally simple automata is in EXSPACE
- Implies that value 1 problem is decidable for structurally simple automata
- Structurally simple automata \subseteq simple automata

Definition of a simple automaton

Definition

Given some infinite word $w \in A^\omega$ and initial distribution α , w induces a simple process if:

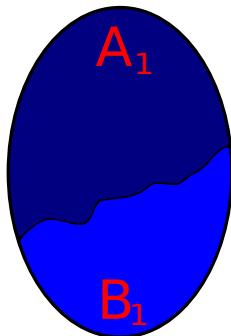
- At every step we can partition Q into two sets: $(A_n)_{n \in \mathbb{N}}$ and $(B_n)_{n \in \mathbb{N}}$ such that:
- exists $\gamma > 0$, s.t. $\forall q \in A_i$ we have $\mathbb{P}(\alpha \xrightarrow{w_{\leq i}} q) > \gamma$ and,
- $\lim_n \mathbb{P}(\alpha \xrightarrow{w_{\leq n}} B_n) = 0$

An automaton where every word induces a simple process is called simple

Definition of a simple automaton

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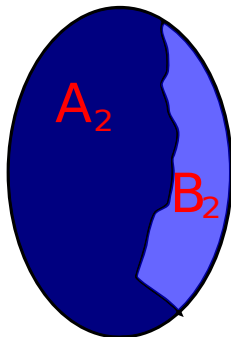
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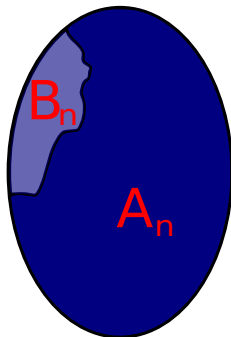
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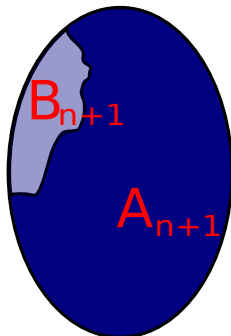
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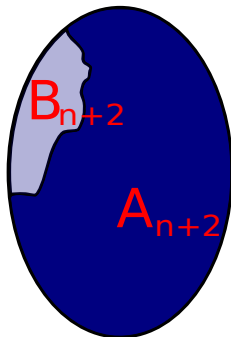
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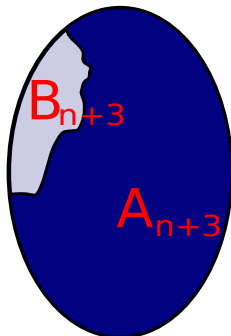
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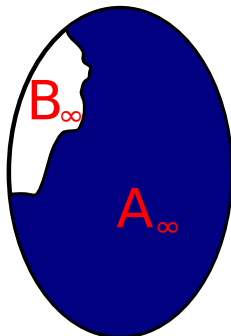
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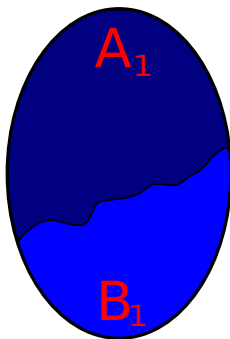
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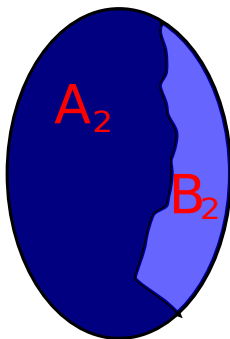
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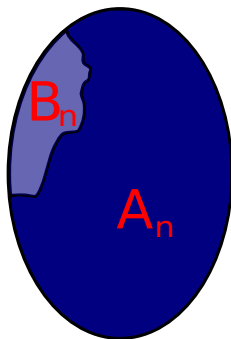
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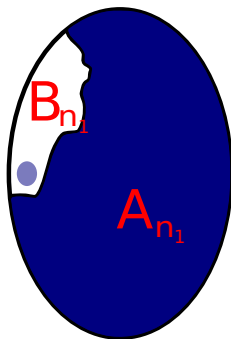
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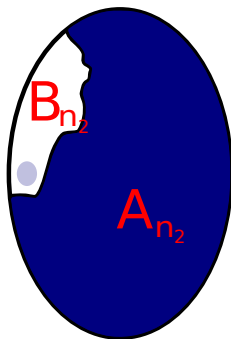
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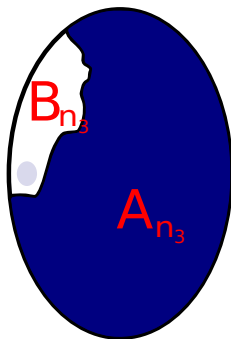
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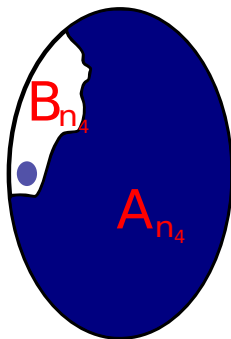
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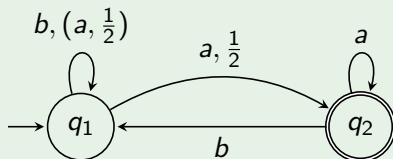
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Example

An example of a non-simple automaton:

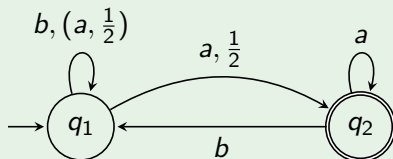
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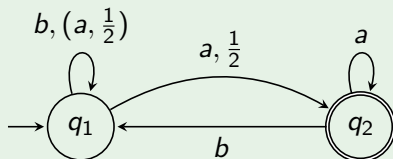


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A sufficient condition for non-simplicity

For a word $w = a_1 a_2 \cdots \in A^\omega$ if there exist states $q, q' \in Q$ and $\gamma > 0$ such that for i.m. n :

- $\mathbb{P}(\alpha \xrightarrow{a_1 a_2 \dots a_n} q) > \gamma$ and,
- $\mathbb{P}(q \xrightarrow{a_{n+1} \dots a_{n+k_n}} q') > 0$ for a $k_n \in \mathbb{N}$.

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Then:

- we can not give a bound for $\mathbb{P}(\alpha \xrightarrow{a_1 \dots a_{n+k_n}} q')$ implies non-simple process

Introduction (Leaktight automata)

- By Fijalkow, Gimbert and Oualhadj, studied in the context of PAs on finite words
- Proof of correctness relies on Simon's factorization forest theorem
- Value 1 problem is decidable for this class
- Membership in this class is decidable
- Both in PSPACE

Markov monoid

Abstract asymptotic behaviour of sequences of finite words $(u_n)_{n \in \mathbb{N}}$ with *limit words*, which are maps from $Q \times Q$ to $\{0, 1\}$. One can

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Definition

Markov monoid is the smallest set containing \mathbf{a} for $a \in A$ and closed under the operations above.

In the extended Markov monoid, we keep track of edges deleted by iteration in second component: $(\mathbf{u}, \mathbf{u}^+)$.

Value 1 witness

Deciding value 1 is done by looking for a *value 1 witness* in the Markov monoid:

Definition

A limit word \mathbf{u} is called a value 1 witness if for all $q \in Q$:

$$\mathbf{u}(i, q) = 1 \implies q \in F$$

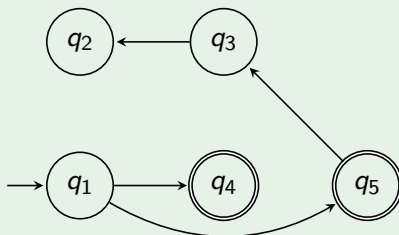
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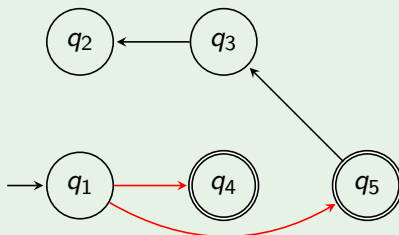
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Example



Algorithm

Require: A probabilistic automaton.

```
1  $\mathcal{G} \leftarrow \{\mathbf{a} \mid a \in A\} \cup \{\mathbf{1}\}$ .
2 repeat
3   if there is  $\mathbf{u}, \mathbf{v} \in \mathcal{G}$  such that  $\mathbf{u} \cdot \mathbf{v} \notin \mathcal{G}$  then
4     add  $\mathbf{u} \cdot \mathbf{v}$  to  $\mathcal{G}$ 
5   if there is  $\mathbf{u} \in \mathcal{G}$  such that  $\mathbf{u}$  is idempotent and  $\mathbf{u}^\# \notin \mathcal{G}$ 
6     then
7       add  $\mathbf{u}^\#$  to  $\mathcal{G}$ 
8   until there is nothing to add
9   if there is a value 1 witness in  $\mathcal{G}$  then
10    return true
11 else
12    return false
```

Leak witness

Similarly membership is decided by looking for a *leak-witness* in the extended Markov monoid:

Definition

An extended limit word $(\mathbf{u}, \mathbf{u}_+)$ is a leak-witness if there are states $r, t \in Q$ such that:

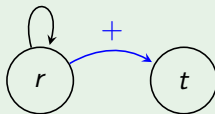
- r is \mathbf{u} -recurrent
- $\mathbf{u}_+(r, t) = 1$
- $\mathbf{u}(t, r) = 0$

Leak witness

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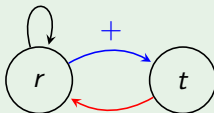


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Leaktight and simple automata

Simple automata

Leaktight

Extended support graph

Markov monoid

Leak witness

Leaktight and simple automata

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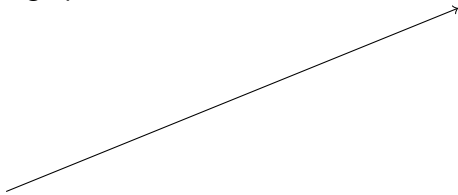
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Non-simplicity witness

Leak witness



Non-simplicity witness

Relation between leaktight and simple automata is in the non-simplicity witness:

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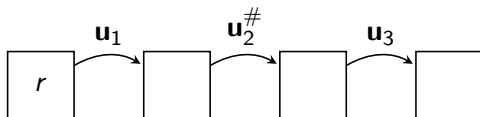
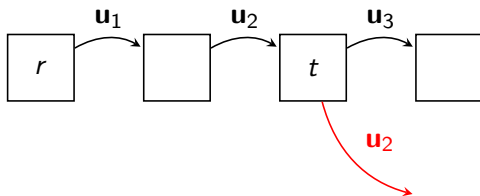
The triple $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ is called a non-simplicity witness if there are states $r, t \in Q$ such that:

- r is $\mathbf{u}_1 \mathbf{u}_2^\# \mathbf{u}_3$ -recurrent
- $\mathbf{u}_1 \mathbf{u}_2(r, t) = 1$
- t is \mathbf{u}_2 -transient

Non-simplicity witness

Definition

- r is $\mathbf{u}_1\mathbf{u}_2^\#\mathbf{u}_3$ -recurrent
- $\mathbf{u}_1\mathbf{u}_2(r, t) = 1$
- t is \mathbf{u}_2 -transient



Simplicity witness implies non-simple process

Proof sketch:

- Concatenate a particular sequence of words that reifies $\mathbf{u}_1 \mathbf{u}_2^\# \mathbf{u}_3$, show that it induces a non-simple process
- Give a bound for distribution on r in i.m. steps,
- Show that we reach t from these points
- Show that t cannot be bounded

\exists leak implies \exists non-simplicity witness

Proof sketch:

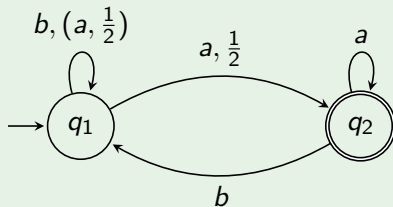
- Look for elements of the extended Markov monoid $(\mathbf{u}, \mathbf{u}_+)$ for which:
- r is \mathbf{u} -recurrent,
- $\mathbf{u}(r, t) = 0$,
- $\mathbf{u}_+(r, t) = 1$

Finally decompose $(\mathbf{u}, \mathbf{u}_+)$ into a triple that is a non-simplicity witness.

Strict inclusion

Inclusion is strict from the example given earlier:

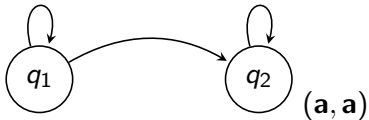
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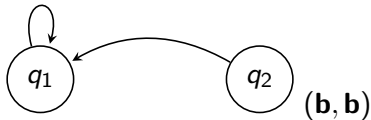
Strict inclusion

But the extended Markov monoid of this automaton contains no leak witness:



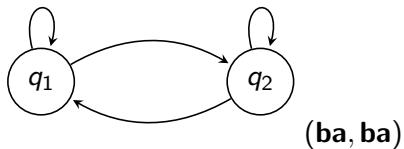
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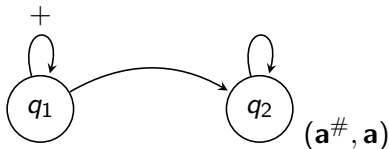
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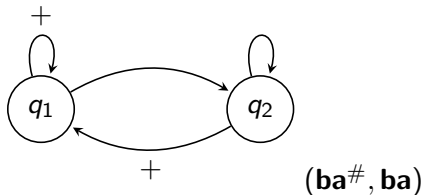
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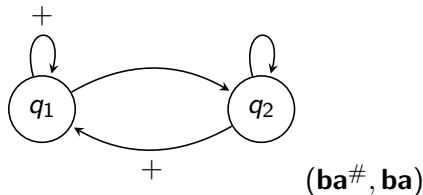
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And it is the monoid defined by these relations:

$$\begin{aligned}
 \mathbf{a}^2 &= \mathbf{a}, & \mathbf{b}^2 &= \mathbf{b}, & \mathbf{ab} &= \mathbf{b}, & \mathbf{aa}^\# &= \mathbf{a}^\#, \\
 \mathbf{a}^{\#2} &= \mathbf{a}^\#, & (\mathbf{ba}^\#)^2 &= \mathbf{ba}^\#, & \mathbf{a}^\# \mathbf{a} &= \mathbf{a}^\#, & \mathbf{a}^\# \mathbf{b} &= \mathbf{b},
 \end{aligned}$$

A conjecture

In the case of leaktight automata, we do qualitative analysis on the PA.

Previous work in quantum automata do quantitative analysis also.

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Previous work in quantum automata do quantitative analysis also.

Derksen, Harm, Emmanuel Jeandel, and Pascal Koiran. "Quantum automata and algebraic groups." *Journal of Symbolic Computation* 39.3 (2005): 357-371.

Corollary: the value 1 problem is decidable for the class of automata with orthogonal matrices.

Conjecture: remains decidable with invertible matrices.

Conclusion and open problems

Leaktight and simple automata were developed with dissimilar methods but $\text{simple} \subset \text{leaktight}$.

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Leaktight and simple automata were developed with dissimilar methods but $\text{simple} \subset \text{leaktight}$.

Problem (open)

Given an isolated cutpoint $0 < \lambda < 1$ decide whether

$$\mathcal{L}_{>}(\mathcal{A}) = \{w \in A^* \mid \mathcal{A}(w) > \lambda\},$$

is empty or not.