Equations for
the polynomial closure

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Classes of languages
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Variety of languages $\mathcal{V}$:

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A \quad \longrightarrow \quad (A^*)\mathcal{V}
$$

alphabet subset of $\text{Rat}(A^*)$

such that

1. $(A^*)\mathcal{V}$ is closed under finite union, finite intersection and complementation.
2. $(A^*)\mathcal{V}$ is closed under quotients: $a^{-1}L, La^{-1} \in (A^*)\mathcal{V}$, for any $L \in (A^*)\mathcal{V}$.
3. if $\varphi: A^* \rightarrow B^*$ is a morphism and $L \in (B^*)\mathcal{V}$, then $L\varphi^{-1} \in (A^*)\mathcal{V}$
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How to characterize algebraically the classes \( \mathcal{V} \) satisfying the following?

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Lattice of languages of $A^*$: set of languages of $A^*$ closed under finite union and finite intersection.

**Theorem (Gehrke, Grigorieff, Pin)**

A set $\mathcal{L}$ of languages of $A^*$ is a lattice of languages closed under quotients if and only if, for some set $\Sigma$ of equations of the form $u \leq v$, with $u, v \in \hat{A}^*$, $\mathcal{L}$ is the set of the languages of $A^*$ that satisfy all equations of $\Sigma$. 
The polynomial closure $\text{Pol}(\mathcal{L})$ of a set of languages $\mathcal{L}$ of $A^*$ is the set of languages that are finite unions of marked products of the form $L_0 a_1 L_1 \cdots a_n L_n$, where the $a_i$ are letters and the $L_i$ are in $\mathcal{L}$. 
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**Examples:**

The polynomial closure of the set of languages of the form $F$ or $F^*$, where $F$ is a finite language is formed by the languages that have star-height one.

The polynomial closure of $\{\emptyset, A^*\}$ is formed by the shuffle ideals of $A^*$ ($u \in L, \ v \in A^* \implies u \sqcup \sqcup v \subseteq L$).
$L_0, L_1, \ldots, L_n \subseteq A^*$, \quad a_1, \ldots, a_n \in A$

$L = L_0 a_1 L_1 \cdots a_n L_n$
Polynomial closure

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Syntactic morphisms:

\[ \eta_i : A^* \rightarrow M(L_i), \quad \mu : A^* \rightarrow M(L) \]

Consider the morphism

\[ \eta : A^* \rightarrow M(L_0) \times \cdots \times M(L_n) \]

\[ u \mapsto (\eta_0(u), \ldots, \eta_n(u)) \]

and the relational morphism

\[ \tau = \eta \mu^{-1} : M(L) \rightarrow M(L_0) \times \cdots \times M(L_n) \]
Proposition (Pin and Weil)

For each idempotent \( e \in M(L_0) \times \cdots \times M(L_n) \), the ordered semigroup \( \tau^{-1}(e) \) satisfies the identity \( x^\omega y x^\omega \leq x^\omega \).
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Theorem (Pin and Weil)

Let \( \mathbb{V} \) be an \( M \)-variety and let \( \mathcal{V} \) be the corresponding variety of languages. Then \( \text{Pol}(\mathcal{V}) \) is a positive variety of languages and the corresponding \( OM \)-variety is the Malcev product \( \llbracket x^\omega y x^\omega \leq x^\omega \rrbracket M \odot \mathbb{V} \).

Theorem (Pin and Weil)

\( \llbracket x^\omega y x^\omega \leq x^\omega \rrbracket M \odot \mathbb{V} \) is defined by the identities \( x^\omega y x^\omega \leq x^\omega \), where \( x, y \in \hat{A}^* \) are such that the identity \( y = x = x^2 \) holds in \( \mathbb{V} \).
Polynomial closure

Let $\mathcal{L}$ be a set of languages of $A^*$.

$\text{Pol}(\mathcal{L})$: 
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Let $\mathcal{L}$ be a set of languages of $A^*$. 

$\text{Pol}(\mathcal{L})$: the set of languages that are finite union of $L_0a_1L_1 \cdots a_nL_n$, with $n \in \mathbb{N}_0$, $L_i \in \mathcal{L}$, $a_j \in A$.

$\Sigma(\mathcal{L})$: the set of equations of the form $x^\omega y x^\omega \leq x^\omega$, where $x, y \in \hat{A}^*$ are such that the equations $y \leq x$ and $x = x^2$ are satisfied by $\mathcal{L}$.
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**Theorem (Branco and Pin)**

If $\mathcal{L}$ is a lattice closed under quotients, then $\text{Pol}(\mathcal{L})$ is defined by $\Sigma(\mathcal{L})$. 
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Idea of proof: later
An example

A regular language is slender iff it is a finite union of languages of the form $u_0 v^* u_1$, where $u_0$, $v$, $u_1$ are words.

Regular slender or full languages of $A^*$ form a lattice of closed under quotients $S$.

The languages of $\text{Pol}(S)$ are finite unions of languages of the form $L_0a_1L_1\cdots a_nL_n$, where the $a_i$ are letters and the $L_i$ are languages of the form $A^*$ or $u^*$ for some word $u$.

**Theorem**

A regular language of $A^*$ belongs to $\text{Pol}(S)$ iff it satisfies the equations of the form

$$(x^\omega y^\omega)^\omega z(x^\omega y^\omega)^\omega \leq (x^\omega y^\omega)^\omega$$

where $z \in A^*$ and $x, y \in A^+$ and $i(x) \neq i(y)$.
An example

Let $L \subseteq A^*$ regular.

$\eta : A^* \rightarrow M(L)$ syntactic morphism.

By last theorem, $L \in \text{Pol}(S)$ iff for any $z \in A^*$ and $x, y \in A^+$ s.t. $i(x) \neq i(y)$,

$$(\hat{\eta}(x)^{\omega} \hat{\eta}(y)^{\omega})^{\omega} \hat{\eta}(z)(\hat{\eta}(x)^{\omega} \hat{\eta}(y)^{\omega})^{\omega} \leq (\hat{\eta}(x)^{\omega} \hat{\eta}(y)^{\omega})^{\omega}.$$ 

Let

$$F = \bigcup_{a, b \in A, a \neq b} \eta(a)M(L) \times M(L)\eta(b)$$

Then

$L \in \text{Pol}(S)$ iff the property $(s^{\omega}t^{\omega})^{\omega} r(s^{\omega}t^{\omega})^{\omega} \leq (s^{\omega}t^{\omega})^{\omega}$ holds for all $(s, t) \in F$, $r \in M(L)$. 

Thus we may conclude:

**Proposition**

*There is an algorithm to decide whether a given regular language belongs to Pol(S).*
Recall the main theorem:

**Theorem (Branco and Pin)**

*If \( \mathcal{L} \) is a lattice closed under quotients, then \( \text{Pol}(\mathcal{L}) \) is defined by \( \Sigma(\mathcal{L}) \).*
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**Theorem (Branco and Pin)**

*If $\mathcal{L}$ is a lattice closed under quotients, then $\text{Pol}(\mathcal{L})$ is defined by $\Sigma(\mathcal{L})$.***

**Proposition**

*If $\mathcal{L}$ is a lattice of languages, then $\text{Pol}(\mathcal{L})$ satisfies $\Sigma(\mathcal{L})$.***
Recall the main theorem:

**Theorem (Branco and Pin)**

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Easier part
Main theorem: idea of proof

Recall:

$\Sigma(\mathcal{L})$: the set of equations of the form $x^\omega y x^\omega \leq x^\omega$, where $x, y \in \hat{A}^*$ are such that the equations $y \leq x$ and $x = x^2$ are satisfied by $\mathcal{L}$. 
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Recall:

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$L \subseteq A^*$ regular.

Define

$$E_L = \left\{ (x, y) \in \widehat{A}^* \times \widehat{A}^* \mid L \text{ satisfies } y \leq x \text{ and } x = x^2 \right\}$$

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**Proposition**

$E_L$ and $F_L$ are clopen in $\hat{A}^* \times \hat{A}^*$. 
Main theorem: idea of proof
Proposition

Let $\mathcal{L}$ be a set of languages of $A^*$ and $K$ be a regular language of $A^*$. TFAE:

1. $K$ satisfies $\Sigma(\mathcal{L})$. 
Main theorem: idea of proof

Proposition

Let $\mathcal{L}$ be a set of languages of $A^*$ and $K$ be a regular language of $A^*$. TFAE:

1. $K$ satisfies $\Sigma(\mathcal{L})$.
2. The set $\{F_K\} \cup \{E_L \mid L \in \mathcal{L}\}$ is an open cover of $\widehat{A^*} \times \widehat{A^*}$. 
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**Proposition**

Let $\mathcal{L}$ be a set of languages of $A^*$ and $K$ be a regular language of $A^*$. If $K$ satisfies $\Sigma(\mathcal{L})$, there exists a finite subset $\mathcal{F}$ of $\mathcal{L}$ such that $K$ satisfies $\Sigma(\mathcal{F})$. 
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Main theorem: idea of proof

An important ingredient:

Simon’s theorem on factorization forests
Main theorem: idea of proof

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Simon’s theorem on factorization forests

Board

Factorization forest: . . .

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